Quantal Andreev Billiards and their Spectral Geometry

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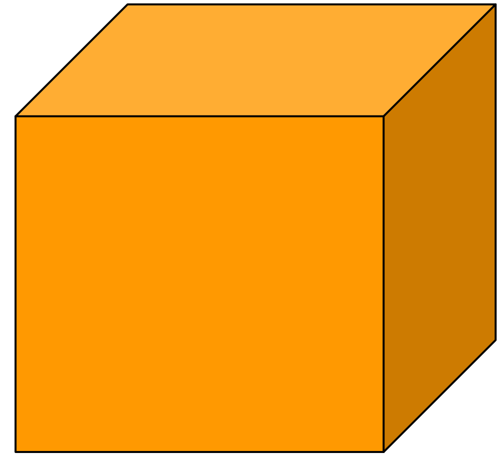
Outline of the talk

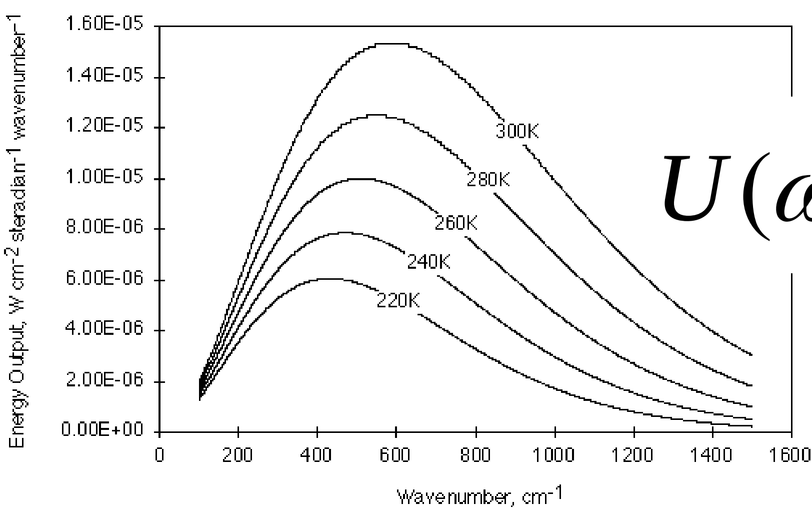
- What is spectral geometry?
- What is Andreev reflection?
- Schrödinger billiards; Andreev billiards
- Andreev billiards: Classical properties
- Andreev billiards: Quantal properties
- Applications to high-Tc superconductors

Planck's BB radiation spectrum

- cubic oven filled with electromag. radiation
- thermal equilibrium
- volume $V >> (\hbar c / \kappa T)^3$
- temperature T
- 2 polariz. states, periodic boundary cond's



$\Rightarrow$ spectral energy density U at frequency ω



$$U(\omega) = \frac{V}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega / \kappa T) - 1}$$

What is spectral geometry?

- Planck
- Lorentz
- Hilbert
- Weyl

What is spectral geometry?

- Planck (1900): introduces quanta to derive black body radiation spectrum

- Lorentz (Wolfskehl Lect. Göttingen1910): suggests short-wave DOS (henceBBRS) depends only on oven *volume* not *shape*

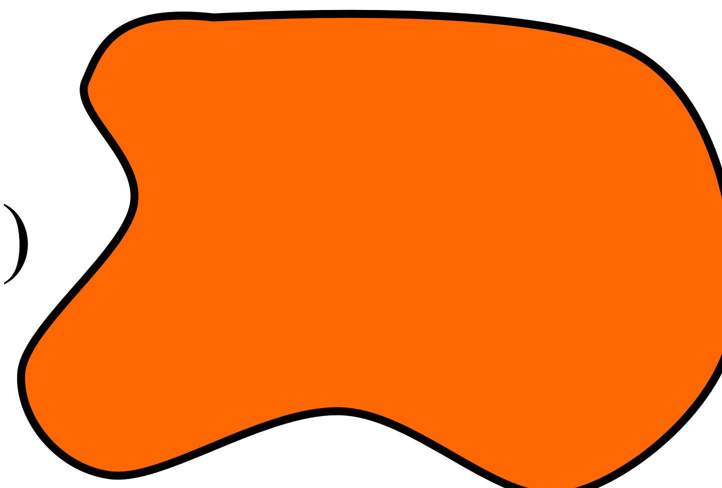
- Hilbert (1910): "…will not be proven in my lifetime…"

- Weyl (1911): Lorentz correct; via Hilbert's integral equation theory

Spectral geometry: 2D Laplacian – "Can one hear the shape of a drum?"

• Kac

- study eigenvalues E of Laplacian:
  $-\nabla^2 \Psi = E\Psi$ with $\Psi = 0$ on the boundary
- origin: energies of Schrödinger particle in box; or normal mode freq's of scalar wave equation
- collect eigenvalues in a distribution (DOS) $\rho(E)$
- study connection of $\rho(E)$ with shape, especially at large E



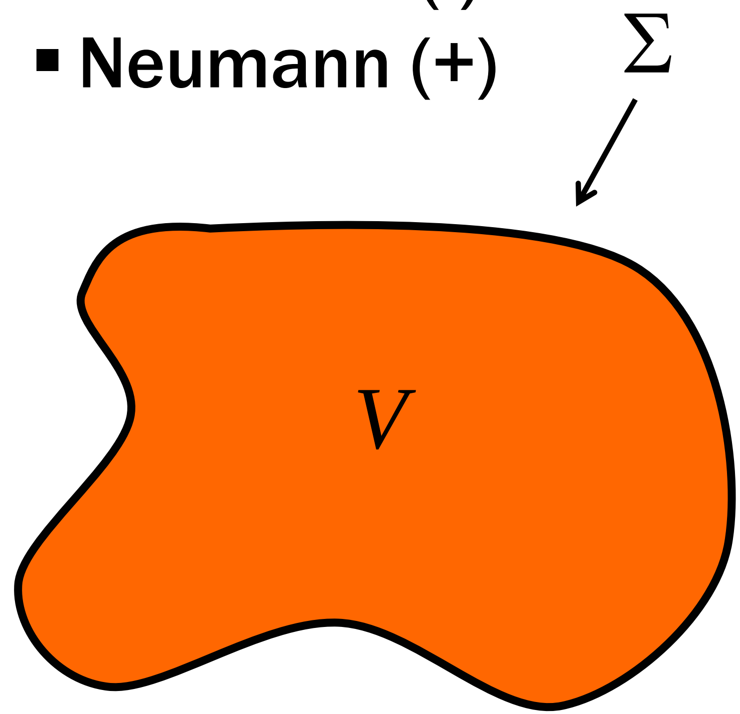
Spectral geometry: 3D Laplacian

– e.g.: smoothed DOS at large energy –

$-\nabla^2 \Psi = E \Psi$ with B.C.s

- Dirichlet (-)
- Neumann (+)

- volume V, surface Σ
- local principal radii of curvature R_1 and R_2



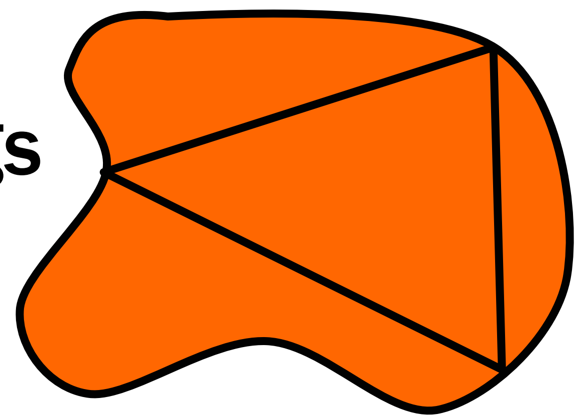
$$\rho(E) \approx \frac{V}{4\pi^2}\sqrt{E} \mp \frac{S}{16\pi}$$

$$+\frac{1}{12\pi^2}\int_\Sigma dS\,\frac{1}{2}\left(R_1^{-1}+R_2^{-1}\right)\frac{1}{\sqrt{E}}+\cdots$$

Spectral geometry: 3D Laplacian

– e.g.: DOS oscillations –

- also gives long-range clustering of levels (semiclass. remnant of quant. shell structure)
- evaluate *MRE* in saddle-point approximation (specular reflection, closed paths,…)
- $\rho(E)$ dominated by classical periodic orbits (cpo), w/ tracings
- trace formulae for DOS osc's (classical ingredients only)



$$\rho_{\mathrm{osc}}(E) \approx \sum_{\mathrm{c.p.o.}\ j} \sum_{\mathrm{trac.}\ p} \frac{A_{jp}}{\hbar^{1+l_j/2}} \sin\{(pS_j(E)/\hbar) + p\alpha_j\}$$

Potential Theory I

- BVP of classical electrostatics
  - solve $\nabla_{\mathbf{r}}^2 \varphi(\mathbf{r}) = 0$
  - with $\varphi(\alpha) = \psi(\alpha)$ on boundary (Dirichlet), or
  - with $\mathrm{n}_\alpha \cdot \nabla \varphi(\alpha) = \psi(\alpha)$ on boundary (Neumann)

- View $\varphi(\mathbf{r})$ as due to charge-layer $\nu(\beta)$ or dipole-layer $\mu(\beta)$ on boundary

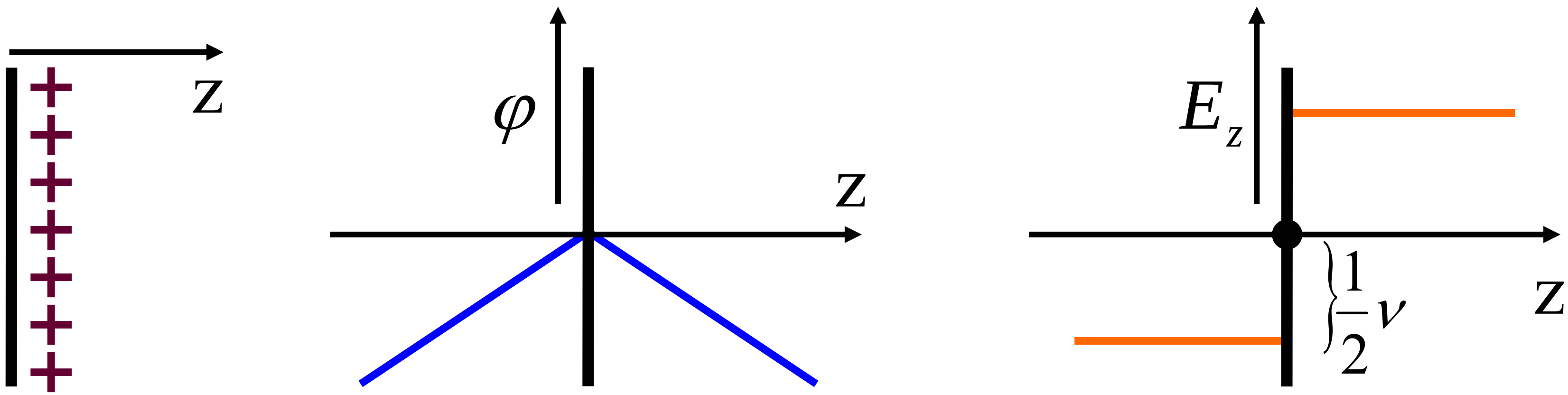
$$\varphi_c(\mathbf{r}) = \int_\Sigma d\sigma_\beta \, G_0(\mathbf{r}, \beta) \, \nu(\beta)$$

$$\varphi_d(\mathbf{r}) = \int_\Sigma d\sigma_\beta \, \mathrm{n}_\beta \cdot \nabla G_0(\mathbf{r}, \beta) \, \mu(\beta)$$

- *G_0 is familiar Coulomb potential*

Potential Theory II

- charge-layer potential φ *continuous* (but electric field E_z *discontinuous*)
- similarly dipole-layer potential *discontinuous*



$$\lim_{r \to \alpha} n_\alpha \cdot \nabla_r \varphi_c(r) = -\frac{1}{2}\nu(\alpha) + \int_\Sigma d\sigma_\beta \, n_\alpha \cdot \nabla_\alpha G_0(\alpha, \beta)\, \nu(\beta)$$

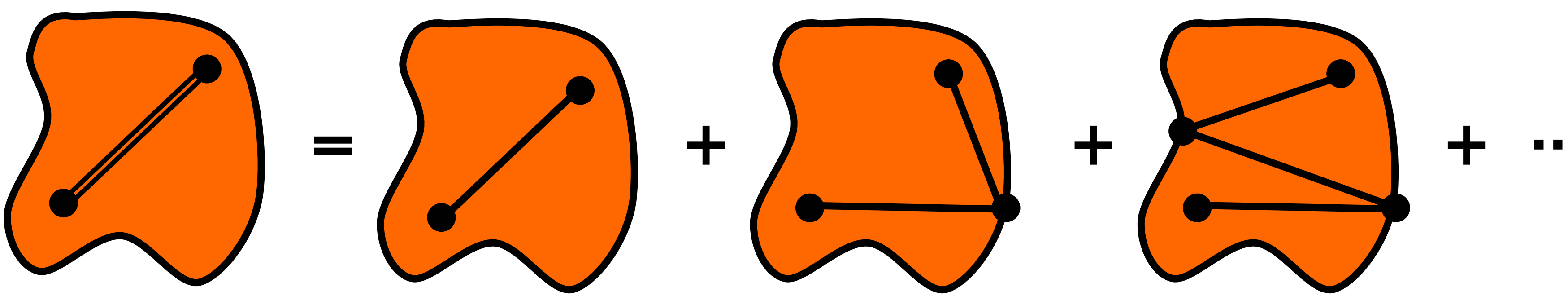
- *apply Neumann BC* $\psi(\alpha)$ *here*

- solve F/II integral equation for ν by iteration

S.G. for Laplacian: Balian-Bloch scheme

- seek Green function $G(\mathbf{r},\mathbf{r}',z)$
- obeys $(-\nabla_{\mathbf{r}}^2 - z)\,G(\mathbf{r},\mathbf{r}',z) = \delta(\mathbf{r}-\mathbf{r}')$

  plus $G(\mathbf{r},\mathbf{r}',z)=0$ for $\mathbf{r}$ on boundary
- use rep. of G from classical potential theory
- obtain G via *Multiple Reflection Expansion*



- evaluate terms via large - E asymptotics
- gives DOS $\rho(E)$ via $\rho(E) = \pi^{-1}\,\mathrm{Im}\,\mathrm{Tr}\,G\,|_{z=E+i0}$

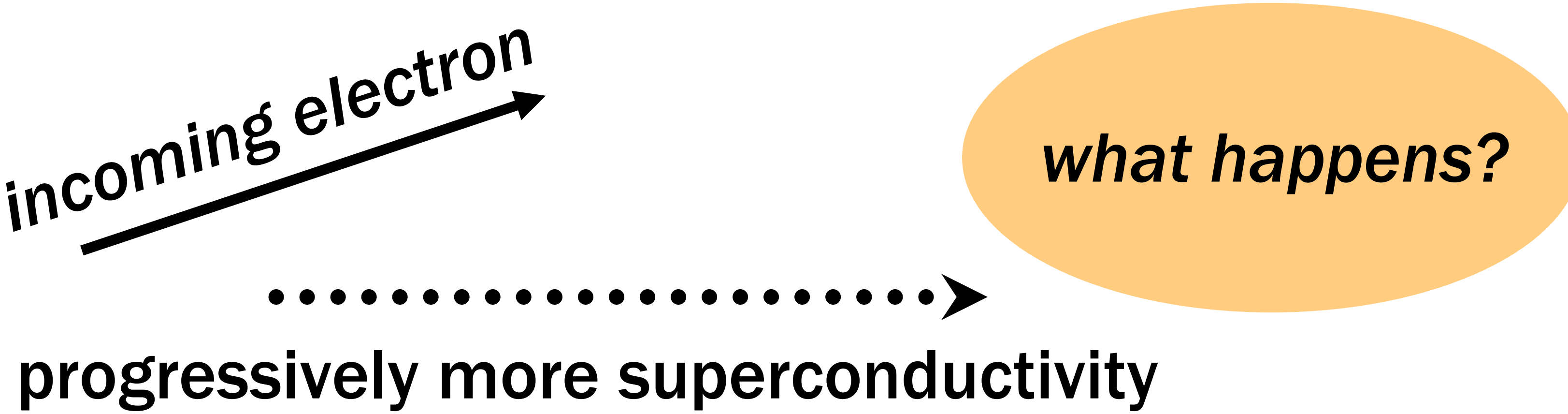
Some further settings for S.G.

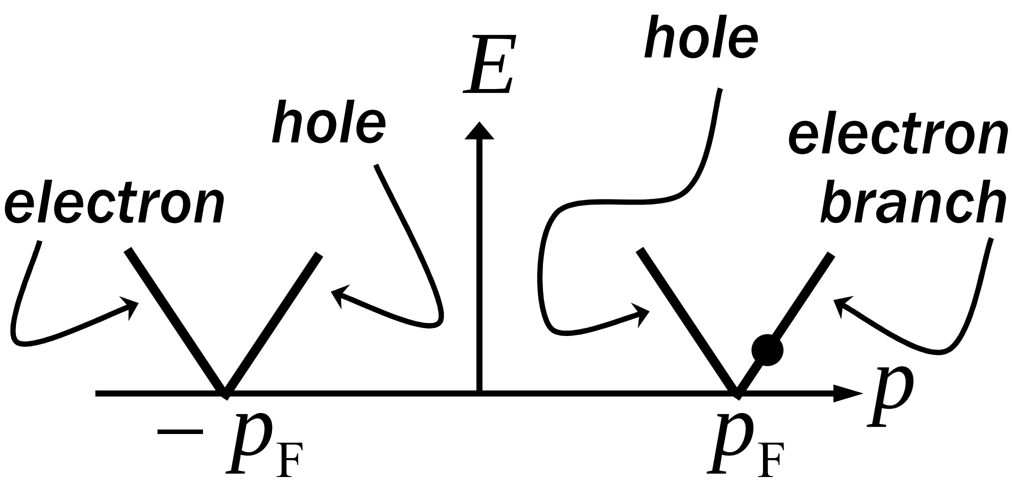
- acoustics and elasticity
- thermodynamics (e.g. electronic, magnetic and vibrational properties of granular matter)
- superfluid films, Casimir effects
- nucleation
- nuclei, atomic clusters, nanoparticles,…

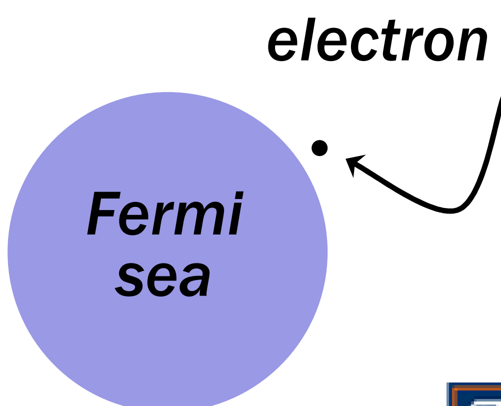
What is Andreev reflection?

[Andreev, Sov. Phys. JETP 19 (1964) 1228]

- low-energy electron quasiparticle approaches superconductor from normal region



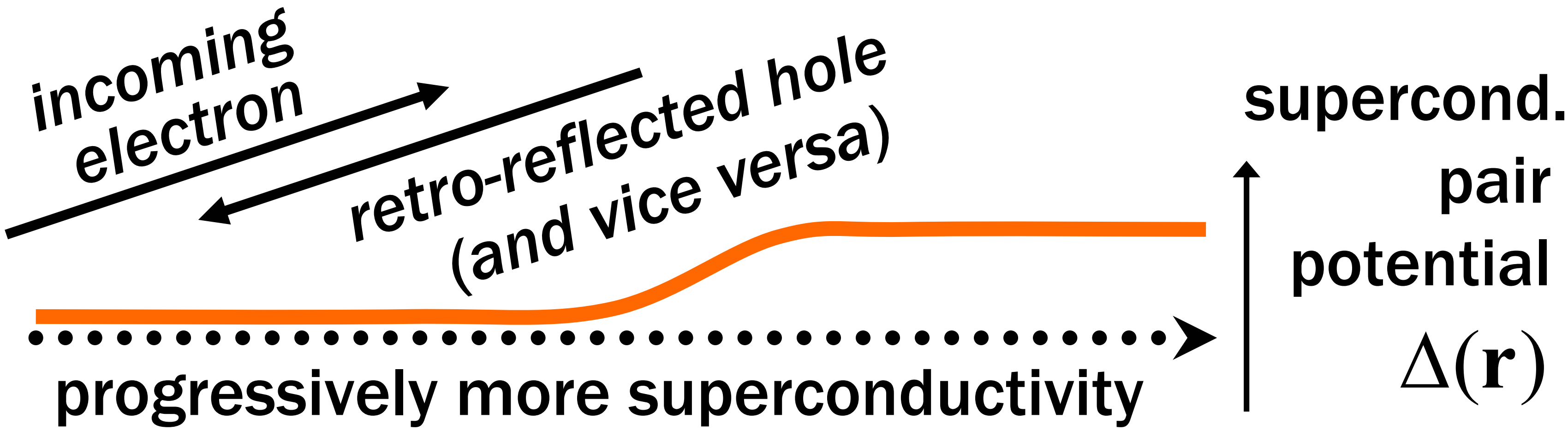


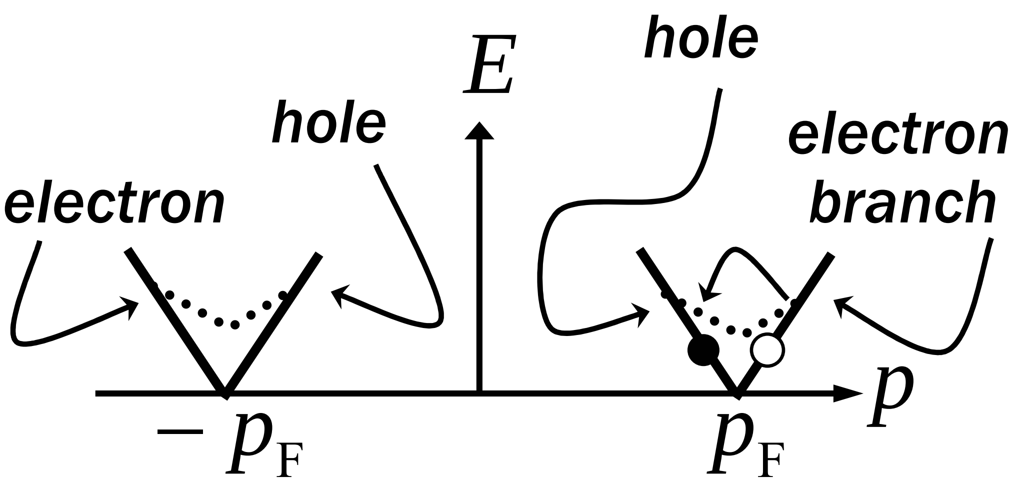


quasiparticle excitation spectrum

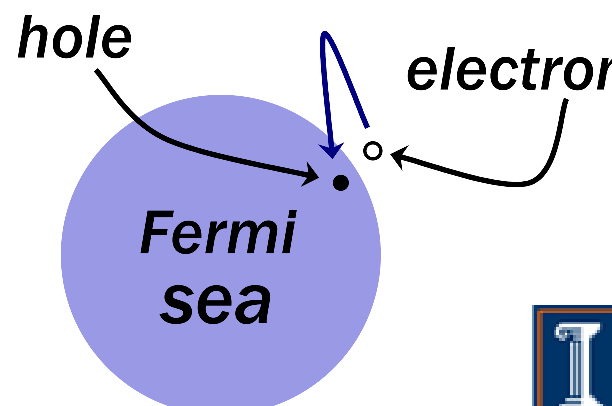
Charge-reversing retro-reflection

- incident low-energy electron quasiparticle retro-reflected as hole quasiparticle





quasiparticle excitation spectrum

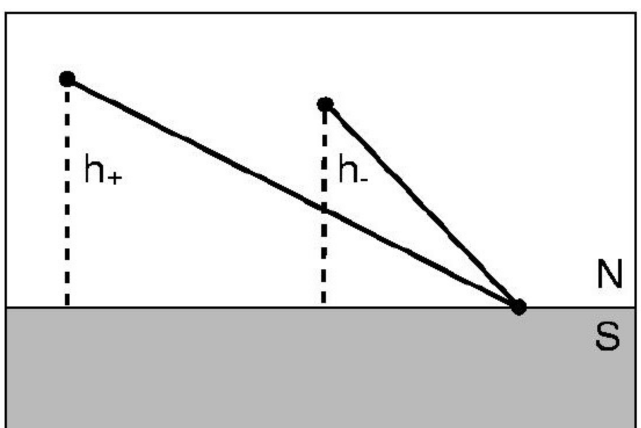


Andreev reflection: semiclassics

- gap excludes single-particle excitations
- velocity must be reversed
- but momentum cannot be...

$$\frac{\delta p}{p_{\mathrm{F}}} \approx \int \frac{dp/dt}{p_{\mathrm{F}}} dt \approx \int \frac{\Delta/\xi}{p_{\mathrm{F}}} dt \approx \frac{\Delta/\xi}{p_{\mathrm{F}}} \frac{\xi}{v_{\mathrm{F}}} \approx \frac{\Delta}{E_{\mathrm{F}}} \approx 10^{-4}$$

- e/h conversion with retro-reflection
- e acquires mate $\Rightarrow$ Cooper-pair + hole

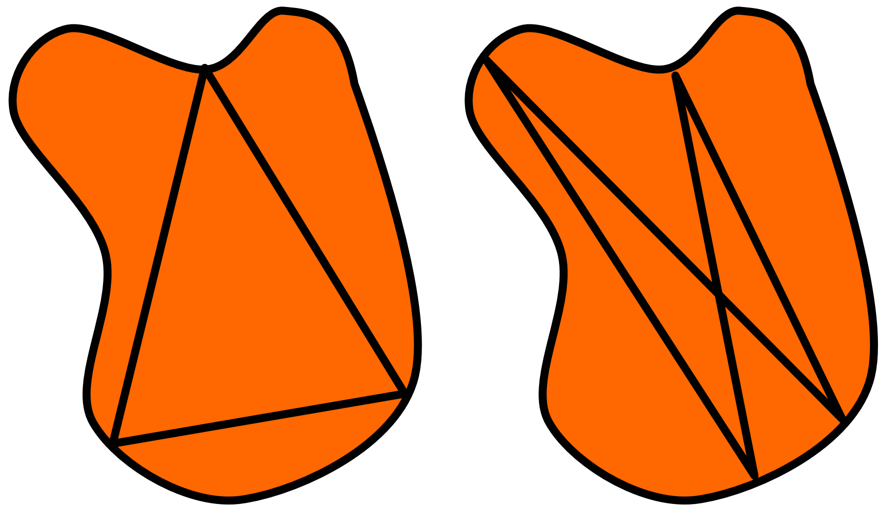


- Snell's law: from action with reversed momentum

Schrödinger billiards

$$-\nabla^2\Psi = (2mE/\hbar^2)\Psi$$

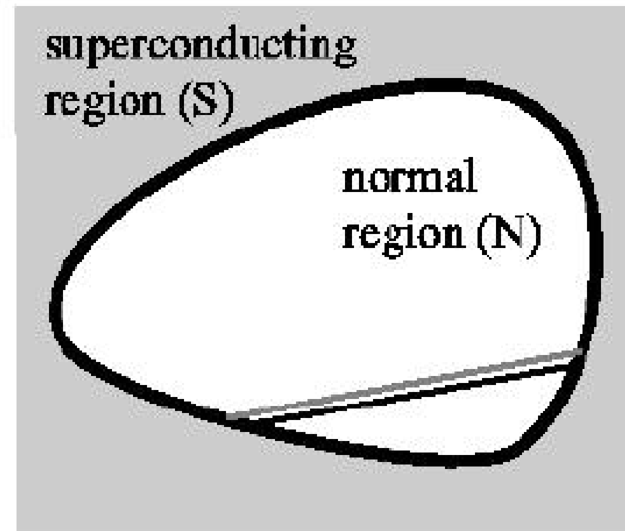
with $\Psi = 0$ on boundary



- quantum mechanics of finite systems
- shape is the only "parameter"
- classical mechanics is "geometry"
- DOS oscillations related to "polygons" via trace formula
- no separation of periods
- quantum implications of classical chaos

Andreev billiards

[Kosztin, Maslov & PMG, Phys. Rev. Lett. 75 (1995) 1735]



What are they?

- normal region surrounded superconductor
- 1-quasiparticle energy gap; energy states confined .
- mechanism: Andreev reflection from boundary

Basic issues

- shape-spectrum connection
  - DOS oscillations related to chordal orbits and creeping orbits via trace formula
  - separation of periods

Andreev billiards: Model system

$$u(\mathbf{r},t)=\langle \Phi_0 | \hat{\Psi}_{\uparrow}(\mathbf{r},t) | \Phi_1 \rangle$$

$$v(\mathbf{r},t)=\langle \Phi_0 | \hat{\Psi}^{+}_{\downarrow}(\mathbf{r},t) | \Phi_1 \rangle$$

- *generic 1-qp state (Heisenberg rep.)*
- *remove spin-up; add spin-down (Heisenberg rep.)*
- *e/h-qp wave functions*
- *ground state (Heis. rep.)*

- one-e/h-qp states:

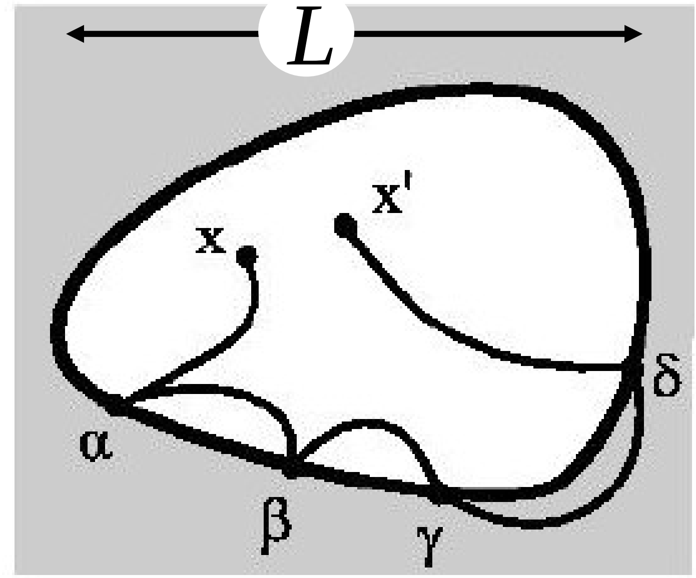
$\Rightarrow$ Bogoliubov-de Gennes eigenproblem

$$\begin{pmatrix} -\nabla^2-\kappa^2 & -\Delta(\mathbf{r}) \\ -\Delta(\mathbf{r})^* & \nabla^2+\kappa^2 \end{pmatrix}\begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix}=E\begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix}$$

where $\frac{\hbar^2}{2m}\left(\kappa^2,E,\Delta\right)=\begin{pmatrix} \text{Fermi} & \text{excitation} & \text{pair} \\ \text{energy} & \text{energy} & \text{potential} \end{pmatrix}$

Andreev billiards: BB scheme

- seek e/h Green function via Balian-Bloch type scheme (potential theory plus *MSE*)
- integrate out propagation in superconductor; yields eff. *<u>M</u>ult. <u>R</u>eflection <u>E</u>xpansion*



- evaluate via two asymptotic schemes

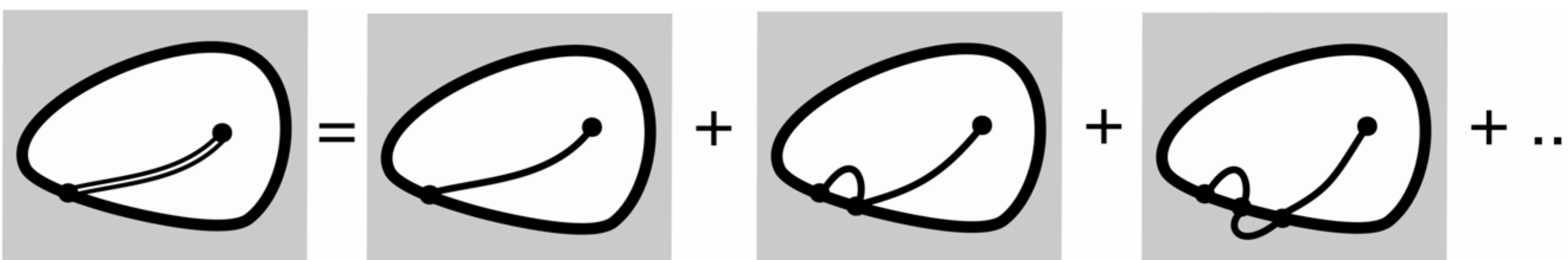
(A) $\kappa L \to \infty,\ \Delta/\kappa^2 \to 0 \quad (L\Delta/\kappa \ \ \text{const.})$

(B) $\kappa L \to \infty,\ \Delta/\kappa^2 \ \text{const.}$

- sets which classical reflections rules hold

Andreev billiards: Effective reflection

- integrate out propagation in superconductor:
  - Separate propagation
    (a) short range
    (b) long range
  - sum all s. r. propagation
    → effective MRE with renormalized Green function
  - l.r. propagation in superconductor vanishes
  - e/h interconversion dominant process



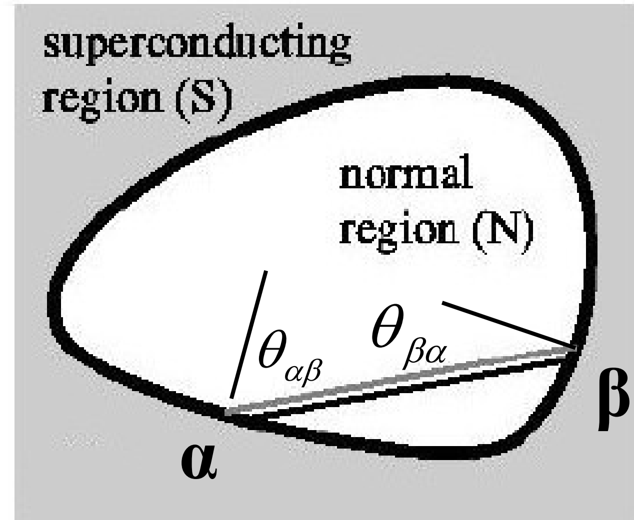
$$G^R = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix} G^N \sim ie^{-i\arccos(E/\Delta)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G^N + O(\Delta/\kappa^2)$$

Normal reflection (r_{ee}, r_{hh}) — Andreev reflection (r_{eh}, r_{he})

Andreev billiards: Scheme A

- limits: $\kappa L \to \infty,\ \Delta/\kappa^2 \to 0 \quad (L\Delta/\kappa \ \text{const.})$
- essentially Andreev
- classical rule:

  perfect retro-reflection



- ladder of states on chords

$$|\boldsymbol{\alpha}-\boldsymbol{\beta}|(E/\kappa)-2\cos^{-1}(E/\Delta)=2\pi n$$

  - *chord length*
  - *e/h momentum difference*
  - *reflection phase-shift*
  - *integer*

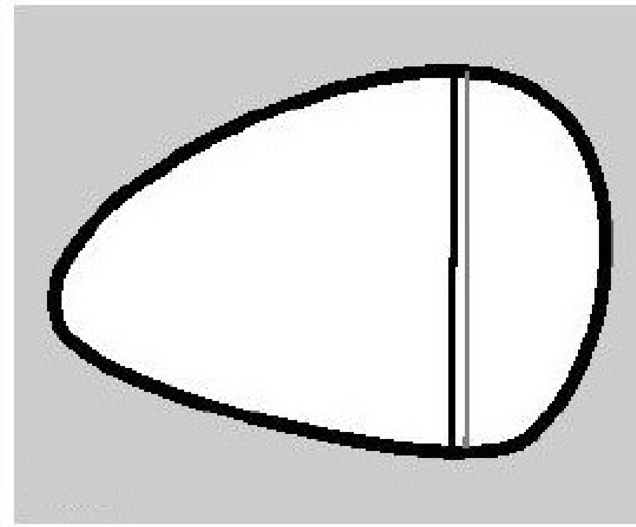
- leads to the scheme-A DOS:

$$\rho(E)\approx \operatorname{Re}\int_{\Sigma}\cos\theta_{\alpha\beta}\cos\theta_{\beta\alpha}\Big/\Big(1-\exp\big\{i|\boldsymbol{\alpha}-\boldsymbol{\beta}|(E/\kappa)-2i\cos^{-1}(E/\Delta)\big\}\Big)$$

Van Hove-type singularities

- **scheme A:** $\kappa L \to \infty, \;\; \Delta/\kappa^2 \to 0 \; (L\Delta/\kappa \;\; \text{const.})$

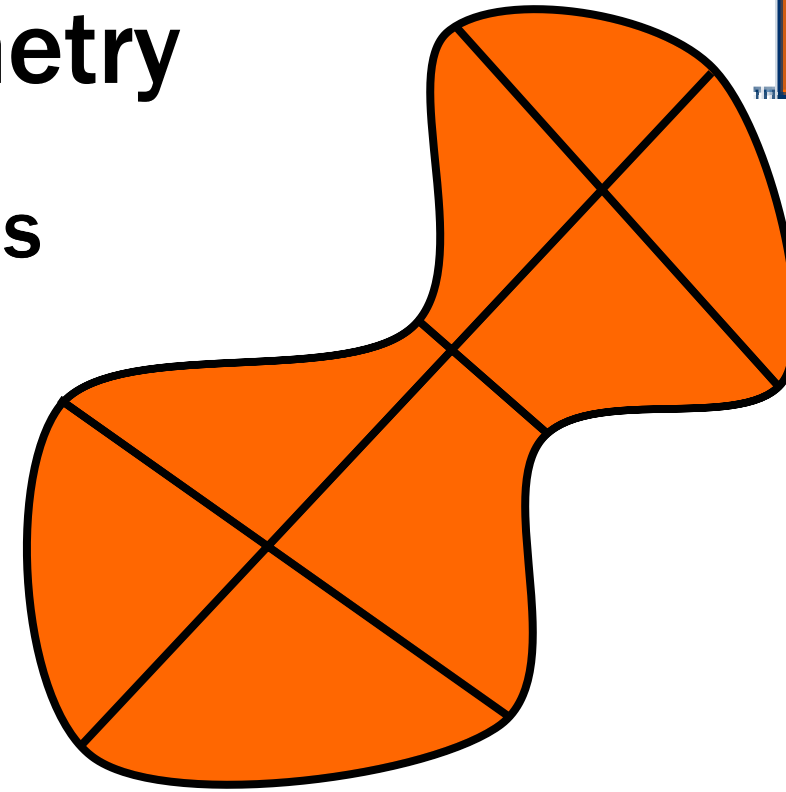
- **role of stationary chords**
  - sharp spectral features
  - signature of bunching of energy levels
  - line-shape “quality” from signature of Hessian
  - quantitative DOS via stationary lengths and end-point curvatures

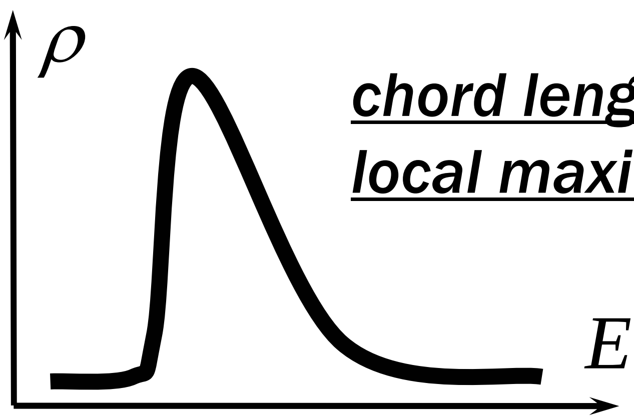


- **can “hear” lengths of the stationary chords**

“Hearing” the geometry

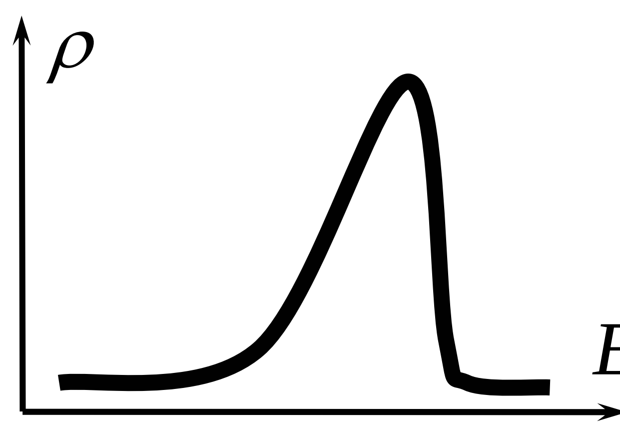
- several stationary chords
- yield distinct features
- some examples…





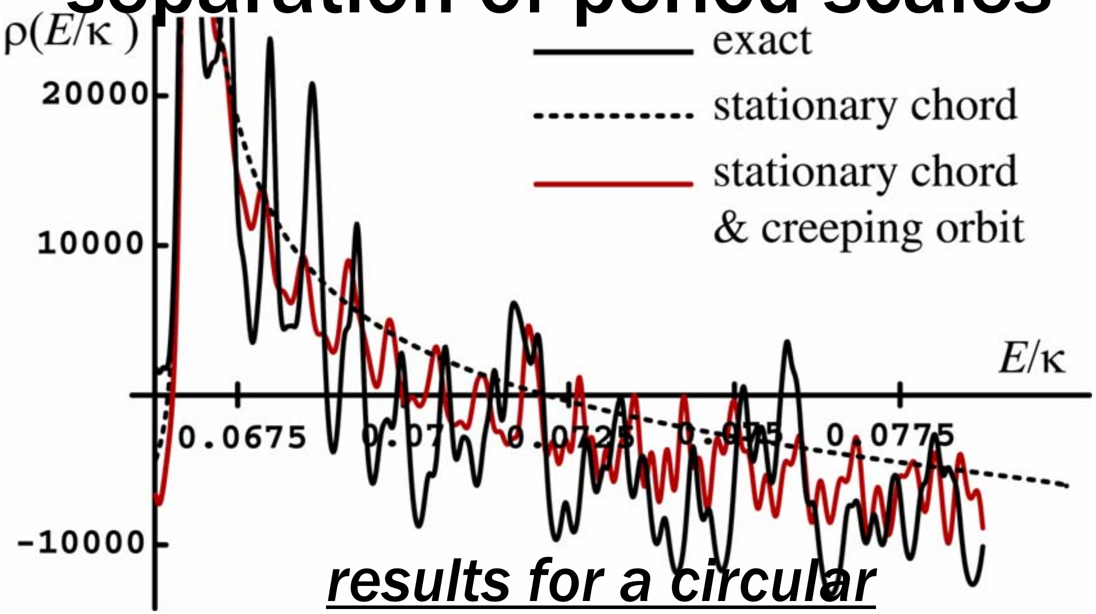
__chord length is local maximum__

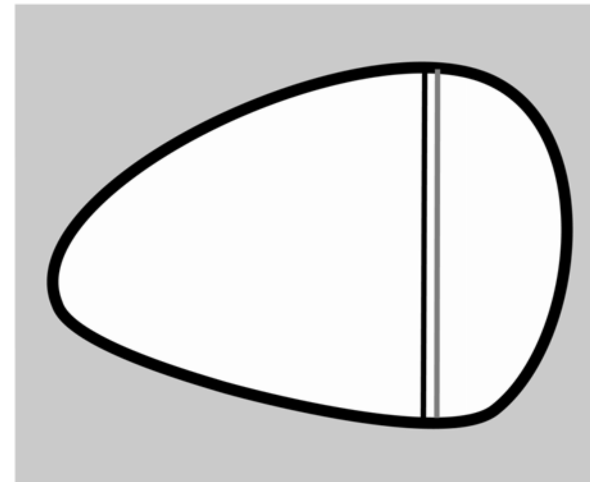
__chord length is local minimum__

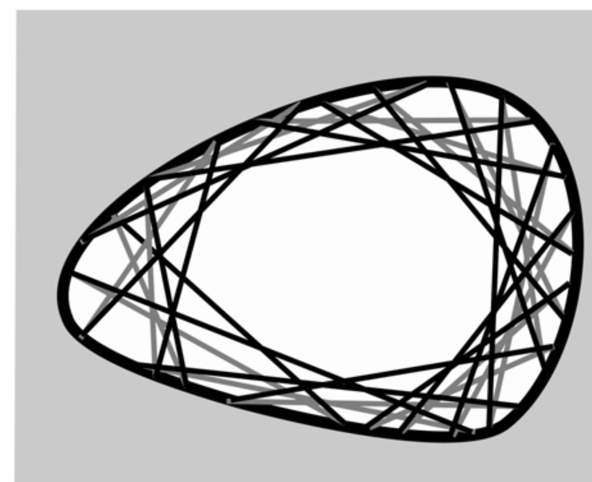


S.G. of Andreev billiards: Scheme B

- scheme B: $\kappa L \to \infty, \; \Delta / \kappa^2$ const.
- retro-refl. no longer perfect
- now class. periodic orbits are
    - stationary chords (coarse)
    - creeping orbits (fine)
- separation of period scales



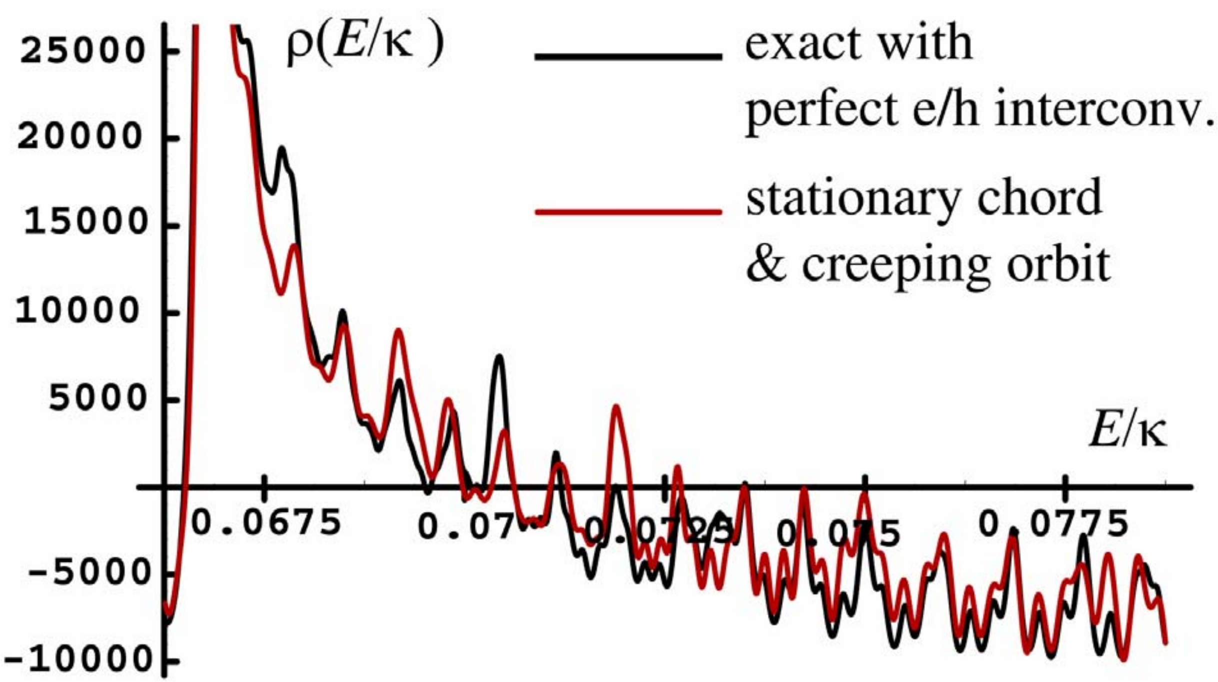




<u>results for a circular Andreev billiard</u>

What's missing?

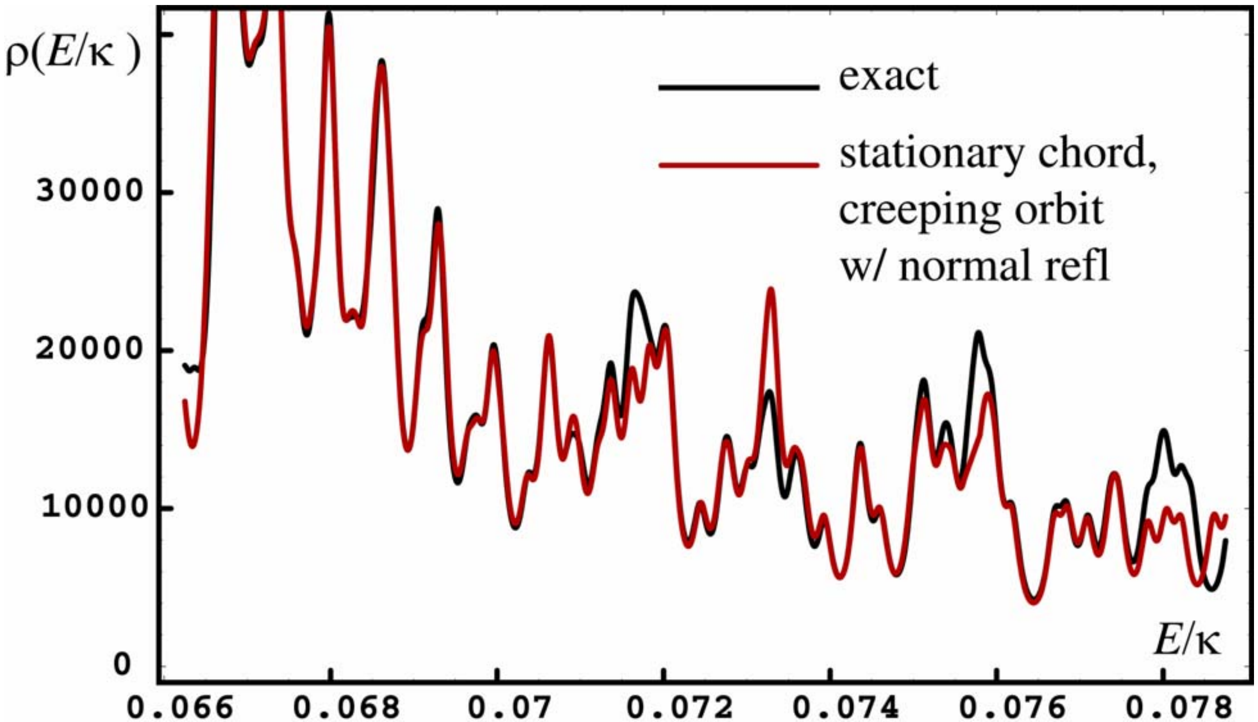
- scheme B: $\kappa L \to \infty,\ \Delta / \kappa^2$ const.
- make *perfect charge-conversion* model
- compare w/ periodic orbit results $\Rightarrow$ *better fit*



$\Rightarrow$ charge-*preserving* processes are missing

Normal reflection in Andreev Billiards

- scheme B: $\kappa L \to \infty,\ \Delta / \kappa^2$ const.
- retro-refl. no longer perfect
- charge interconv. no longer perfect



Some conclusions

- Quantal properties of Andreev billiards
  - discussed in terms of spectral geometry
- Basic structures
  - Andreev's approximation (all chords)
  - fine structure needs creeping orbits etc.
- "Can "hear" novel geometrical features
  - distribution of chord lengths
  - stationary chords, degeneracies and curvatures
- Still to do…
  - self-consistent superconductivity
  - soft and ray-splitting Schrödinger billiards
  - making an Andreev billiard!

A few references

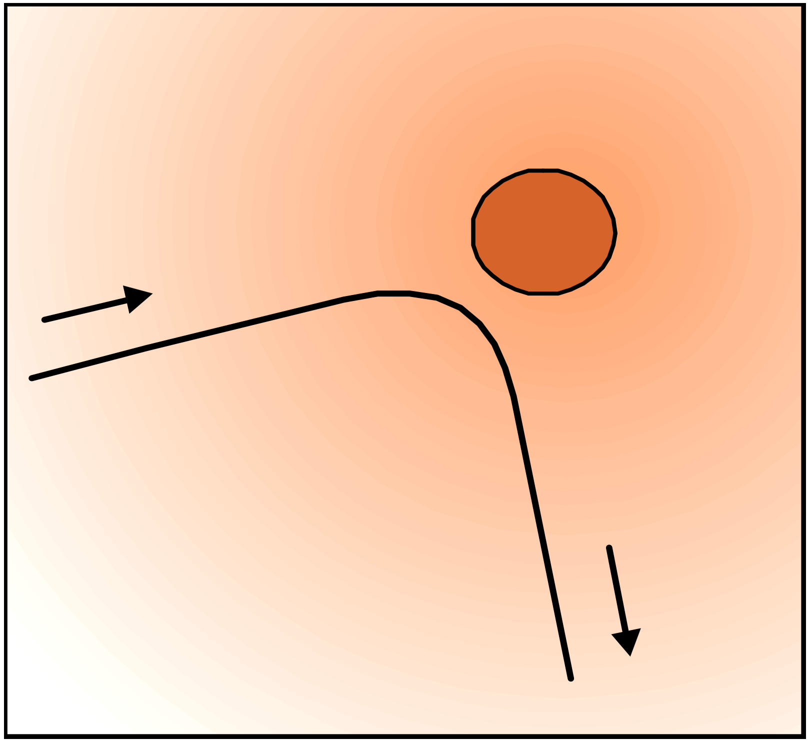
- Andreev, Sov. Phys. JETP 19 (1964) 1228
- Abrikosov, *Fundamentals of the theory of metals*
- Kac, Am. Math. Monthly 73 (1966) 1
- Balian & Bloch, Ann. Phys. (NY) 60 (1970) 401
- Balian & Bloch, Ann. Phys. (NY) 69 (1972) 76
- Baltes & Hilf, *Spectra of finite systems* (1976)
- Guenther & Lee, *PDEs of mathematical physics and integral equations* (Dover, 1996; Sec. 8-7)
- Berry, in *Chaotic behaviour of deterministic systems* (Les Houches 1981; ed. by Iooss et al.)
- Kosztin, Maslov & PMG, Phys. Rev. Lett. 75 (1995) 1735
- Adagideli & PMG, Phys. Rev. B 65 (2002) 201306(R); Int. J. Mod. Phys. B (2002, in press)

Side-products I: Impurity states in d-wave supercondutor

- nonmagnetic impurity
- semiclassical motion in presence of impurity

$$\mu\ddot{\mathbf{x}}_c(s) = -\nabla V(\mathbf{x}_c)$$

- quantum-mechanical electron-hole scattering along classical trajectory



$$\begin{pmatrix} -2ik_F\partial_s & \Delta \\ \Delta^* & 2ik_F\partial_s \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} u \\ v \end{pmatrix}$$

- along trajectory Δ determined by classical position and velocity: $\Delta \equiv \Delta\left(\mathbf{x}_c(s); k_F\dot{\mathbf{x}}_c(s)\right)$

Side-products I: Impurity states in d-wave supercondutor

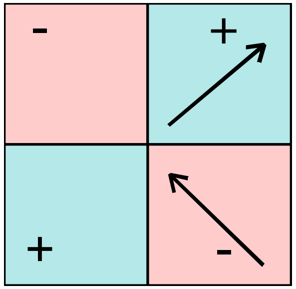
- a realization of Witten's supersymmetric quantum mechanics

- focus on E=0: $(2k_{\mathrm{F}}\partial_s \pm \Delta)\varphi_{\mp} = 0$

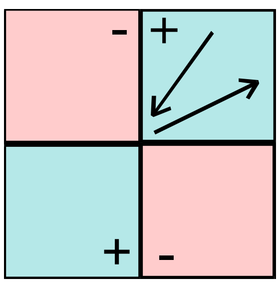
$$\Rightarrow \varphi_{\pm} = \exp\left(\pm\int ds\Delta(s)/2k_{\mathrm{F}}\right)$$

- normalizability $\Rightarrow$ low-energy states for trajectories on which Δ changes sign

Impurity states in d-wave supercon.

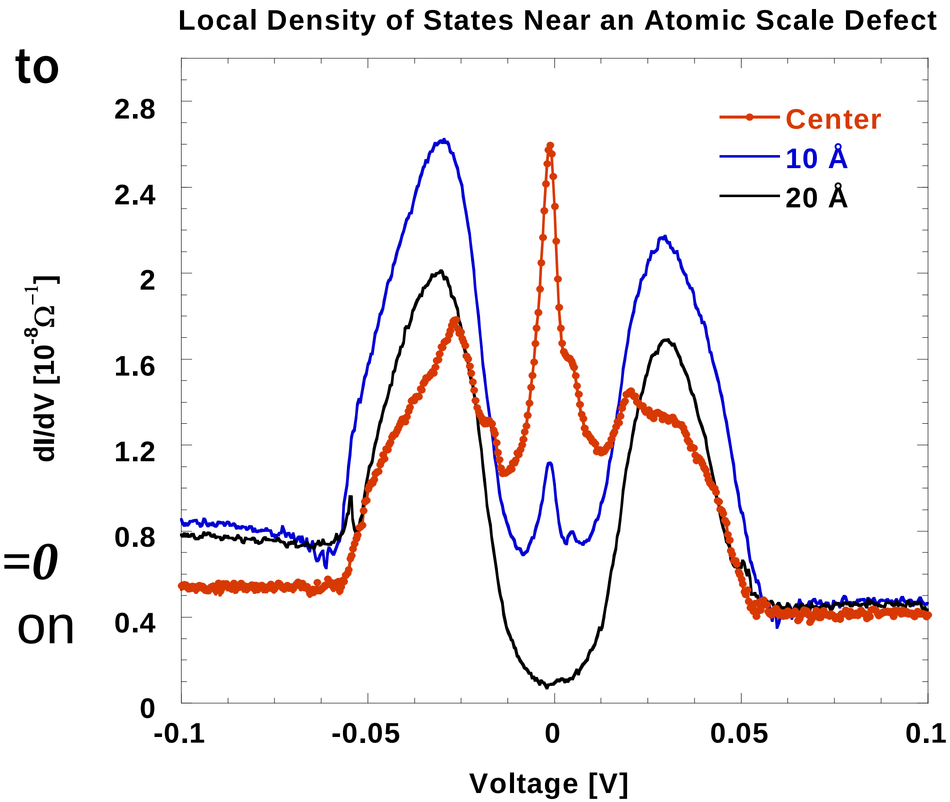


- gives state at $E=0$
- degeneracy proportional to linear extent of impurity
- no dep. on details of Δ



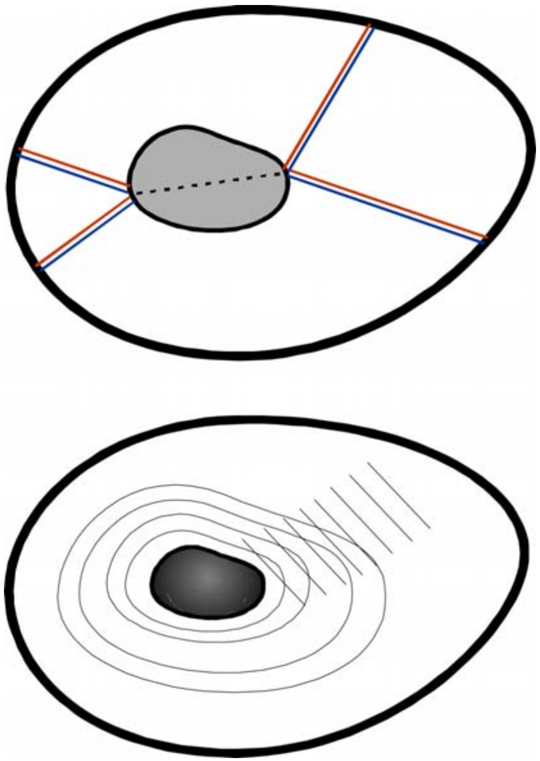
- does not give state at $E=0$
- shape of DOS depends on details of Δ

Yazdani et al.; Hudson et al. (1999)



$$\rho = \rho_{BG} + \delta(E)\, k_F R \sum_{\substack{\text{over all} \\ \text{s/c trajectories}}} \begin{cases} 1, \text{if sign change in } \Delta \\ 0, \text{if no sign change} \end{cases}$$

Side products I: Impurity in d-wave



- tunneling through the impurity
- diffraction effects in scattering

→ • transition between zero energy states

→ • splitting of E=0 peak

Side products II: Imp. in pseudogap