Lattice models and their symmetries

Systems of interest in i) statistical physics

ii) high energy physics

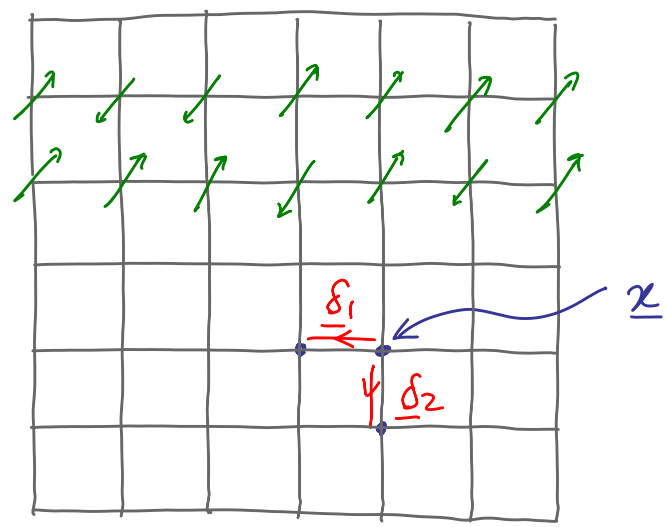
See, e.g., A.M. Polyakov, Gauge Fields and Strings

1 Lattice models having discrete global symmetries

Example: Ising model of ferromagnetism in two dimensions

Sites: vertices $\underline{x}$ of a square lattice

freedoms: $\sigma_{\underline{x}} = \pm 1$ at each $\underline{x}$



Energy: $$\mathcal{E}[\sigma_{\underline{x}}] = -J \sum_{(\underline{x},\underline{\delta})} \sigma_{\underline{x}} \, \sigma_{\underline{x}+\underline{\delta}}$$

Coupling strength (exchange interaction)

Covers all nearest neighbor pairs

Ising spins

$\mathcal{E}$ favors spin alignment, especially at large J / low temp

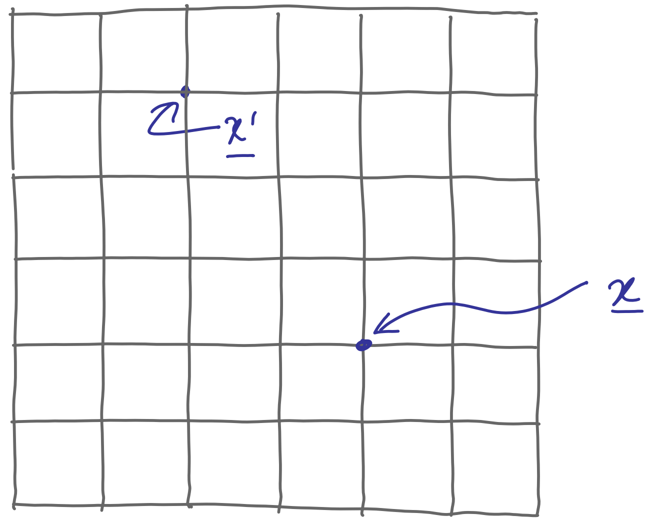
Discrete global symmetry: $\mathbb{Z}_2$

Under $\sigma_{\underline{x}} \to -\sigma_{\underline{x}}$ (for all $\underline{x}$), $\mathcal{E} \to \mathcal{E}$

So "spin-flipped" configurations have identical Boltzmann weights.

Lattice models having continuous global symmetries

Sites: vertices $\underline{x}$ of a square lattice



$g \in$ group G

Freedoms: $\underline{\underline{D}}(g(\underline{x}))$ for each $\underline{x}$

matrix representation of G

Energy: $$\mathcal{E}[\underline{\underline{D}}] = -\sum_{\underline{x},\underline{x}'} J_{\underline{x},\underline{x}'} \operatorname{Tr} \underline{\underline{D}}(g(\underline{x})^{-1}) \cdot \underline{\underline{D}}(g(\underline{x}'))$$

full double sum

site-site coupling

Continuous global symmetry:

$$g(\underline{x}) \to u\, g(\underline{x})\, v \qquad (u, v \in G)$$

so the weight is invariant under $G \otimes G$ transformations

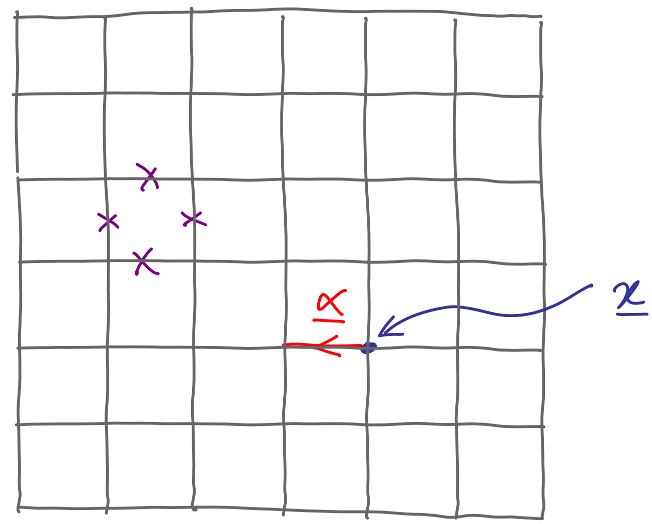
We also have models in which the variables are drawn not from groups but from coset spaces; e.g., vector Heisenberg models:

N-component vector

$$\mathcal{E} = -J \sum_{(\underline{x},\underline{\delta})} \underline{S}(\underline{x}) \cdot \underline{S}(\underline{x}+\underline{\delta})$$

invariant under global rotations of the spins

3 Lattice models having discrete local (a.k.a. gauge) symmetries



Now attach variables $\sigma = \pm 1$ not to the sites but to the links

$$(\underline{x}, \underline{\alpha})$$

beginning of link — direction of link

Take for the energy

$$\mathcal{E} = -\sum_{x\,\underline{\alpha}\,\underline{\beta}} \sigma_{\underline{x},\underline{\alpha}}\, \sigma_{\underline{x}+\underline{\alpha},\underline{\beta}}\, \sigma_{\underline{x}+\underline{\alpha}+\underline{\beta},-\underline{\alpha}}\, \sigma_{\underline{x}+\underline{\beta},-\underline{\alpha}}$$

product of link variables taken around the plaquette

include each plaquette once

Remarkable property:

a) ascribe a sign $\eta_{\underline{x}} = \pm 1$ to every site $\underline{x}$

b) transform the link variables using the η's at its ends

$$\sigma_{\underline{x}\,\underline{\alpha}} \to \eta_{\underline{x}}\, \sigma_{\underline{x}\,\underline{\alpha}}\, \eta_{\underline{x}+\underline{\alpha}}$$

c) Observe that $\mathcal{E}$ is invariant under this enormous group comprising one Z_2 for every lattice site.

Note the huge ground state degeneracy: $\sigma_{\underline{x}\,\underline{\alpha}} = \eta_{\underline{x}}\, \eta_{\underline{x}+\underline{\alpha}}$ (called a "pure gauge" configuration)

We can also make continuous and nonabelian versions of gauge theories; and we can couple them to matter (i.e, spin-like) fields.

In the continuum limit, which has important subtleties associated with the topology of the spaces from which the fields are drawn, one obtains continuum field theories having local symmetries.

For systems possessing global symmetries the various phases of the systems can commonly be diagnosed via a local order parameter (eg the spontaneous magnetization).

Such order parameters necessarily vanish in locally-symmetric theories, and their phases are instead diagnosed by what are called Wilson loops

$$W(C) = \langle \prod \text{link variables} \rangle$$

closed path; ordered product

weighted average over configurations

and how they scale with the shape (perimeter? area?) of the loop.