We can now decompose D into irreducible representations; the result is

$$
\begin{aligned}
D &= \tfrac{1}{4}\left[1 + 2\cos\frac{m\pi}{2} + \cos m\pi\right][D^{(1)} + D^{(2)}] \\
&\quad + \tfrac{1}{4}\left[1 - 2\cos\frac{m\pi}{2} + \cos m\pi\right][D^{(3)} + D^{(4)}] \\
&\quad + \tfrac{1}{2}[1 - \cos m\pi]D^{(5)} \\
&= \begin{cases} D^{(1)} + D^{(2)} & m = 4, 8, 12, \ldots \\ D^{(3)} + D^{(4)} & m = 2, 6, 10, \ldots \\ D^{(5)} & m = 1, 3, 5, 7, \ldots \end{cases}
\end{aligned}
\tag{16-50}
$$

Thus, if m is *even*, the representation is reducible and is the sum of two one-dimensional representations, so that the perturbation can split the degeneracy. If, on the other hand, m is *odd*, the representation is irreducible and no splitting is possible since the symmetry connects all states.

Notice that in this example we have drawn *qualitative* conclusions from group theory; the *amounts* of splitting due to the perturbation are not explored. This is a general feature of group-theoretic arguments. However, we shall now do a problem where we actually calculate numbers by using group theory.

Consider three point masses m, at the vertices of an equilateral triangle, connected by springs of spring constant k (see Figure 16–2). What are the normal modes of this mechanical system? We suppose the masses are only allowed to move in the plane of the page.

We shall number the masses as shown in the figure. Let the coordinates of m_1 relative to its equilibrium position be x_1, y_1, and similarly for the other two masses. We represent the configuration of the system by a six-dimensional "state vector" ξ. In an "elementary coordinate system," such as the one on page 155, ξ has components

$$\xi = (x_1, y_1, x_2, y_2, x_3, y_3) \tag{16-51}$$

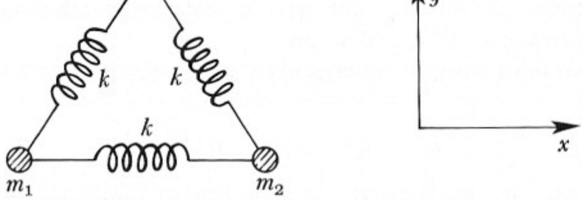

Figure 16–2 The vibrating triangle

The kinetic and potential energies of our system are [compare Eqs. (6-58) and (6-60)]

$$
\begin{aligned}
T &= \tfrac{1}{2}m \sum_i \dot{\xi}_i^2 \\
V &= \tfrac{1}{2}k\{(x_2 - x_1)^2 + [-\tfrac{1}{2}(x_3 - x_2) + \tfrac{1}{2}\sqrt{3}(y_3 - y_2)]^2 \\
&\quad + [\tfrac{1}{2}(x_1 - x_3) + \tfrac{1}{2}\sqrt{3}(y_1 - y_3)]^2\} \\
&= \tfrac{1}{2}k \sum_{ij} V_{ij}\xi_i\xi_j
\end{aligned}
\tag{16-52}
$$

Newton's second law gives

$$m\ddot{\xi}_i = -\frac{\partial V}{\partial \xi_i} = -k \sum_j V_{ij}\xi_j \tag{16-53}$$

For vibration in a normal mode, we will have $\xi \sim e^{-i\omega t}$. Then (16-53) becomes

$$\sum_j V_{ij}\xi_j = \lambda\xi_i \qquad \text{where} \qquad \lambda = \frac{m\omega^2}{k} \tag{16-54}$$

The normal modes are the eigenvectors of the matrix V, with the eigenvalues giving the frequencies.

Let us see what group theory can do to find these eigenvalues. In the first place, each eigenvector generates an irreducible representation when we act on it with all the elements of our symmetry group. Thus, in a coordinate system which diagonalizes V,

$$V = \begin{pmatrix} \lambda_a & & & & & \\ & \ddots & & & & \\ & & \lambda_a & & & \\ & & & \lambda_b & & \\ & & & & \ddots & \\ & & & & & \lambda_b \\ & & & & & & \text{etc.} \end{pmatrix} \begin{matrix} \Big\} D^{(a)} \\ \\ \Big\} D^{(b)} \end{matrix} \tag{16-55}$$

The first n_a of our new coordinate vectors are eigenvectors belonging to the eigenvalue λ_a, and transforming among each other according to the irreducible representation $D^{(a)}$, the next n_b eigenvectors belong to λ_b and transform according to $D^{(b)}$, and so on.

As usual, we must discuss the symmetry group briefly. It consists of six elements

$$I \qquad R \qquad R^2 \qquad P \qquad PR \qquad PR^2$$

where I is the identity, R rotates the triangle 120° in a positive sense (counterclockwise), and P reflects about a vertical line through the center. The character table is shown in Table 16-8. This symmetry group is just S_3; compare Tables 16-1 and 16-3.

When various group elements act on the triangle, they induce linear transformations of the ξ_i. For example, if R operates on the triangle,

$$\xi' = D(R)\xi \tag{16-56}$$

where the matrix $D(R)$ is

$$D(R) = \begin{pmatrix} 0 & 0 & r \\ r & 0 & 0 \\ 0 & r & 0 \end{pmatrix} \qquad r = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix} \tag{16-57}$$

Similarly,

$$D(P) = \begin{pmatrix} 0 & p & 0 \\ p & 0 & 0 \\ 0 & 0 & p \end{pmatrix} \qquad p = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16-58}$$

What representations does this representation D contain? Its character is

$$\chi(C_1) = 6 \qquad \chi(C_2) = 0 \qquad \chi(C_3) = 0$$

and this gives [compare (16-32) and (16-33)]

$$D = D^{(1)} \oplus D^{(2)} \oplus 2D^{(3)} \tag{16-59}$$

Note that D is just (equivalent to) the regular representation of our symmetry group. This particular feature is an accident, and not very general.

Now we can be more specific and write, in a coordinate system with the eigenvectors of V as basis,

$$V = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & 0 & \\ & & \lambda_{31} & & & \\ & & & \lambda_{31} & & \\ & 0 & & & \lambda_{32} & \\ & & & & & \lambda_{32} \end{pmatrix} \begin{matrix} D^{(1)} \\ D^{(2)} \\ \left.\right\} D^{(3)} \\ \\ \left.\right\} D^{(3)} \\ \\ \end{matrix} \tag{16-60}$$

where we have indicated the transformation properties of the various eigenvectors. Note that the two pairs of eigenvectors transforming like $D^{(3)}$ need not have the same eigenvalue.

Table 16-8

Class	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$
$C_1(I)$	1	1	2
$C_2(R, R^2)$	1	1	-1
$C_3(P, PR, PR^2)$	1	-1	0

What does the matrix $D(G)V$, where G is an arbitrary element of our symmetry group, look like in this special coordinate system? The answer is clearly

$$D(G)V = \begin{pmatrix} \lambda_1 D^{(1)}(G) & & & \\ & \lambda_2 D^{(2)}(G) & & \\ & & \lambda_{31} D^{(3)}(G) & \\ & & & \lambda_{32} D^{(3)}(G) \end{pmatrix} \tag{16-61}$$

This is not directly useful, of course, because we do not know what this coordinate system is; that is, we do not know the eigenvectors. However, the trace of this matrix is invariant to coordinate transformations, so that

$$\text{Tr } D(G)V = \lambda_1 \chi^{(1)}(G) + \lambda_2 \chi^{(2)}(G) + (\lambda_{31} + \lambda_{32})\chi^{(3)}(G) \tag{16-62}$$

in *any* coordinate system.

Now it is straightfoward to compute the following traces by making use of the specific forms (16-52), (16-57), and (16-58) of V, $D(R)$, and $D(P)$, respectively.

$$\begin{aligned} \text{Tr } D(I)V &= 6 \\ \text{Tr } D(R)V &= \tfrac{3}{2} \\ \text{Tr } D(P)V &= 3 \end{aligned} \tag{16-63}$$

Our eigenvalues therefore obey the equations

$$\begin{aligned} \lambda_1 + \lambda_2 + 2(\lambda_{31} + \lambda_{32}) &= 6 \\ \lambda_1 + \lambda_2 - (\lambda_{31} + \lambda_{32}) &= \tfrac{3}{2} \\ \lambda_1 - \lambda_2 &= 3 \end{aligned}$$

from which

$$\lambda_1 = 3 \qquad \lambda_2 = 0 \qquad \lambda_{31} + \lambda_{32} = \tfrac{3}{2} \tag{16-64}$$

We could determine λ_{31} and λ_{32} separately by looking at things like

$$\text{Tr } V^2 = \lambda_1^2 + \lambda_2^2 + 2(\lambda_{31}^2 + \lambda_{32}^2)$$

but a simpler way is to note that there must be three degrees of freedom having zero eigenvalue, two translational and one rotational. Thus

$$\lambda_{31} = 0 \qquad \lambda_{32} = 3/2 \tag{16-65}$$

and we have determined all the eigenvalues without solving any secular equation.