For this integral $\rho(s) = s - \ln s$. Thus, $\rho'(s) = 1 - 1/s$ and $\rho' = 0$ when $s = 1$. So there is a simple (second-order) saddle point at $s = 1$.

To ascertain the structure of the saddle point we let $s = u + iv$ and identify the real and imaginary parts of ρ: $\rho(s) = u - \ln \sqrt{u^2 + v^2} + i(v - \arctan v/u)$. At $s = 1$, $\rho = 1$. Therefore, paths of constant phase (steepest curves) emerging from $s = 1$ must satisfy

$$v - \arctan v/u = 0.$$

There are two solutions to this equation: $v = 0$ and $u = v \cot v$. These two curves are shown on Fig. 6.12. In Prob. 6.64 you are asked to verify that (*a*) the steepest-descent curves are correctly shown on Fig. 6.12; (*b*) as s moves away from $s = 1$, steepest-descent curves emerge from $s = 1$ initially parallel to the Im $s = v$ axis; (*c*) the steepest-descent curves cross the v axis at $\pm i\pi/2$ and approach $s = -\infty \pm i\pi$.

To use the method of steepest descents, we simply shift the contour C so that it is just the steepest-descent contour on Fig. 6.12 which passes through the saddle point at $s = 1$. Let us review why we choose such a contour. In general, we always choose a steepest-descent contour because on such a contour we can apply the techniques of Laplace's method directly to complex integrals. If the steepest-descent contour is finite and does not pass through a saddle point, then the maximum value of $|e^{x\rho}|$ must occur at an endpoint of the contour and we need only perform a local analysis of the integral at this endpoint. However, in the present example the contour has no endpoint and is infinitely long. It is crucial that it pass through a saddle point because $|e^{x\rho}|$ reaches its maximum at the saddle point and decays exponentially as $s \to \infty$ along *both* of the steepest-descent curves. If there were no saddle point, then, although $|e^{x\rho}|$ would decrease in one direction along the contour, it would increase in the other direction and the integral would not even converge!

Now we proceed with the asymptotic expansion of the integral in (6.6.23). We can approximate the steepest-descent contour in the neighborhood of $s = 1$ by the straight line $s = 1 + iv$. This gives the Laplace integral

$$\frac{1}{\Gamma(x)} \sim \frac{1}{2\pi x^{x-1}} \int_{-\varepsilon}^{\varepsilon} dv\, e^{x(1 - v^2/2)}, \qquad x \to +\infty,$$

which we evaluate by letting $\varepsilon \to \infty$:

$$\frac{1}{\Gamma(x)} \sim \frac{1}{2\pi x^{x-1}} \frac{e^x}{\sqrt{x}} \sqrt{2\pi}, \qquad x \to +\infty.$$

We thereby recover the result in (6.6.21).

Example 10 *Steepest-descents approximation of a real integral where Laplace's method fails.* In this example we consider the real integral

$$I(x) = \int_0^1 dt\, e^{-4xt^2} \cos(5xt - xt^3) \tag{6.6.24}$$

in the limit $x \to +\infty$. This integral is *not* a Laplace integral because the argument of the cosine contains x. Nonetheless, one might think that one could use the ideas of Laplace's method to approximate the integral. To wit, one would argue that as $x \to +\infty$, the contribution to the integral is localized about $x = 0$. Thus, a very naive approach is simply to replace the argument of the cosine by 0. If this reasoning were correct, then we would conclude that

$$I(x) \sim \int_0^1 dt\, e^{-4xt^2} \sim \sqrt{\frac{\pi}{16x}}, \qquad x \to +\infty. \qquad \text{(WRONG)}$$

This result is clearly incorrect because e^{-xt^2} does not become exponentially small until t is larger than $1/\sqrt{x}$. Thus, when $t \sim 1/\sqrt{x}$ $(x \to +\infty)$, the argument of the cosine is *not* small. In particular, the term $5xt$ is large and the cosine oscillates rapidly. This suggests that there is destructive interference and that $I(x)$ decays much more rapidly than $\sqrt{\pi/16x}$ as $x \to +\infty$.

Can we correct this approach by including the $5xt$ term but neglecting the xt^3 term? After all, when t lies in the range from 0 to $1/\sqrt{x}$, the term $xt^3 \to 0$ as $x \to +\infty$. Thus, xt^3 does not even shift the phase of the cosine more than a fraction of a cycle. If we were to include just the $5xt$ term, we would obtain

$$\begin{aligned} I(x) &\sim \int_0^1 dt\, e^{-4xt^2} \cos(5xt), \qquad x \to +\infty, \\ &\sim \int_0^\infty dt\, e^{-4xt^2} \cos(5xt), \qquad x \to +\infty, \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-4xt^2 + 5ixt} \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-x(2t-5i/4)^2 - 25x/16} \\ &= \tfrac{1}{4}\sqrt{\pi/x}\, e^{-25x/16}, \qquad x \to +\infty. \qquad \text{(WRONG)} \end{aligned}$$

Although this result is exponentially smaller than the previous wrong result, it is also wrong! It is incorrect to neglect the xt^3 term (see Prob. 6.65).

But if we cannot neglect even the xt^3 term, then how can we make any approximation at all? It should not be necessary to do the integral exactly to find its asymptotic behavior!

The correct approach is to use the method of steepest descents to approximate the integral at a saddle point in the complex plane. To prepare for this analysis we rewrite the integral in the following convenient form:

$$\begin{aligned} I(x) &= \tfrac{1}{2}\int_{-1}^{1} dt\, e^{-4xt^2 + 5ixt - ixt^3} \\ &= \tfrac{1}{2}e^{-2x}\int_{-1}^{1} dt\, e^{x\rho(t)}, \end{aligned} \tag{6.6.25}$$

where

$$\rho(t) = -(t-i)^2 - i(t-i)^3. \tag{6.6.26}$$

Our objective now is to find steepest-descent (constant-phase) contours that emerge from $t = 1$ and $t = -1$, to distort the original contour of integration t: $-1 \to 1$ into these contours, and then to use Laplace's method. To find these contours we substitute $t = u + iv$ and identify the real and imaginary parts of ρ:

$$\begin{aligned} \rho(t) &= \phi + i\psi \\ &= -v^3 + 4v^2 - 5v + 3u^2v - 4u^2 + 2 + i(3uv^2 - 8uv + 5u - u^3). \end{aligned} \tag{6.6.27}$$

Note that the phase of $\psi = \operatorname{Im}\rho$ at $t = 1$ and at $t = -1$ is different: $\operatorname{Im}\rho(-1) = -4$, $\operatorname{Im}\rho(1) = 4$. Thus, there is no single constant-phase contour which connects $t = -1$ to $t = 1$.

Our method is similar to that used in Examples 1 and 2. We follow steepest-descent contours C_1 and C_2 from $t = -1$ and from $t = 1$ out to ∞. Next, we join these two contours at ∞ by a third contour C_3 which is also a path of constant phase. C_3 must pass through a saddle point because its endpoints lie at ∞; otherwise, the integral along C_3 will not converge (see the discussion in Example 9).

There are two saddle points in the complex plane because $\rho'(t) = -2(t-i) - 3i(t-i)^2 = 0$ has two roots, $t = i$ and $t = 5i/3$. The contour C_3 happens to pass through the saddle point at i. On Fig. 6.13 we plot the three constant-phase contours C_1, C_2, and C_3. It is clear that the original contour C can be deformed into $C_1 + C_2 + C_3$. (In Prob. 6.66 you are to verify the results on Fig. 6.13.)

The asymptotic behavior of $I(x)$ as $x \to +\infty$ is determined by just three points on the contour $C_1 + C_2 + C_3$: the endpoints of C_1 and C_2 at $t = -1$ and at $t = +1$ and the saddle point at i. However, the contributions to $I(x)$ at $t = \pm 1$ are exponentially small compared with

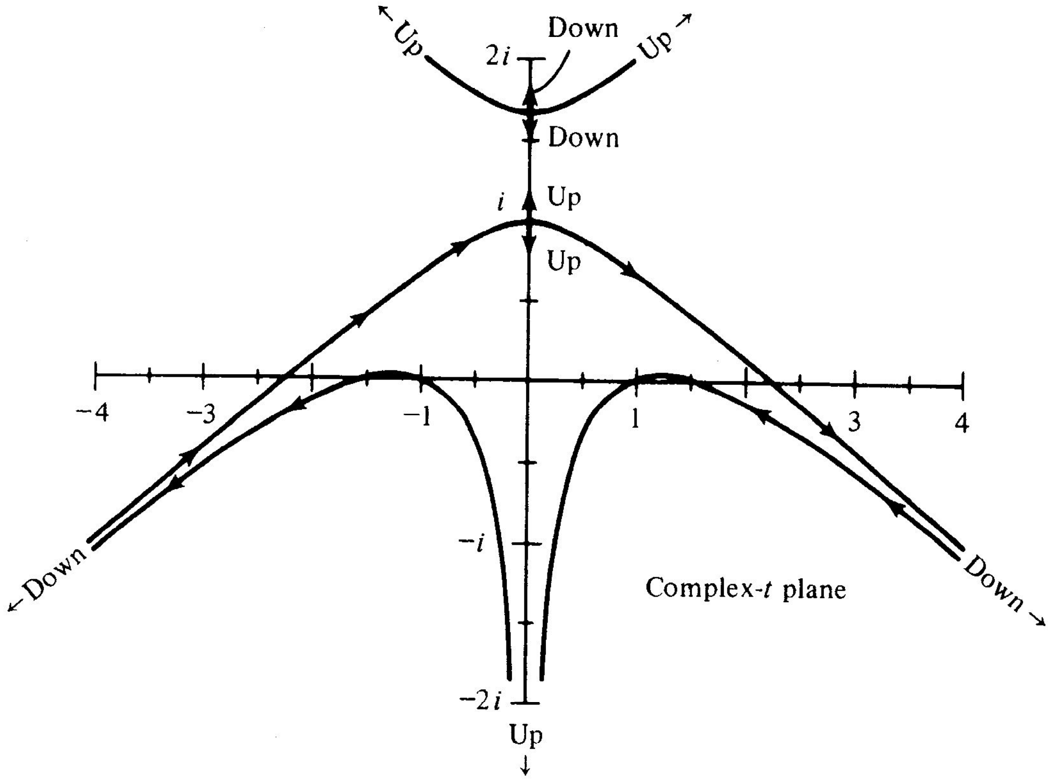

Figure 6.13 To approximate the integral in (6.6.25) by the method of steepest descents we deform the original contour connecting the points $t = -1$ to $t = 1$ along the real axis into the three distinct steepest-descent contours above, one of which passes through a saddle point at $t = i$. Steepest-ascent and -descent curves near a second saddle point at $t = 5i/3$ and steepest-ascent curves going from 1 and -1 to $-i\infty$ are also shown, but these curves play no role in the calculation.

that at $t = i$ (see Prob. 6.67). Near $t = i$ we can approximate the contour C_3 by the straight line $t = i + u$ and $\rho(t)$ by $\rho(t) \sim -u^2$ $(u \to 0)$. Thus,

$$I(x) \sim \tfrac{1}{2}e^{-2x}\int_{-\varepsilon}^{\varepsilon} e^{-xu^2}\,du, \qquad x \to +\infty,$$

$$\sim \tfrac{1}{2}e^{-2x}\sqrt{\pi/x}, \qquad x \to +\infty. \tag{6.6.28}$$

This, finally, is the correct asymptotic behavior of $I(x)$! This splendid example certainly shows the subtlety of asymptotic analysis and the power of the method of steepest descents.

Example 11 *Steepest-descents analysis with a third-order saddle point.* In Example 5 of Sec. 6.5 and Prob. 6.55 we showed that

$$J_x(x) \sim \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)x^{-1/3}, \qquad x \to +\infty. \tag{6.6.29}$$

Here we rederive (6.6.29) using the method of steepest descents.

We begin with the complex-contour integral representation for $J_\nu(x)$:

$$J_\nu(x) = \frac{1}{2\pi i}\int_C dt\, e^{x\sinh t - \nu t}, \tag{6.6.30}$$

where C is a Sommerfeld contour that begins at $+\infty - i\pi$ and ends at $+\infty + i\pi$. Setting $\nu = x$ gives

$$J_x(x) = \frac{1}{2\pi i}\int_C dt\, e^{x(\sinh t - t)}. \tag{6.6.31}$$