1) Uncorrelated (or unentangled) states: Two identical bosons or fermions are in the normalised state

$$|\Psi\rangle = A \sum_{ij} c_i \, d_j \, a_i^\dagger \, a_j^\dagger \, |0\rangle,$$

where i labels an orthonormal set of single-particle states, $\{c_i\}$ and $\{d_i\}$ are complex constants parametrising $|\Psi\rangle$, and $\{a_i^\dagger\}$ and $\{a_i\}$ are creation and annihilation operators for the single-particle states. Such a state is said to be uncorrelated (or unentangled) except for the effect of statistics.

a) Suppose that the complex numbers $\{c_i\}$ and $\{d_i\}$ satisfy

$$\sum_i |c_i|^2 = \sum_i |d_i|^2 = 1.$$

Determine the normalisation constant A in terms of the sum $S = \sum_i c_i^* \, d_i$.

b) Compute the expectation value, in the state $|\Psi\rangle$, of a one-particle operator $\sum_a f^{(a)}$ in terms of the constants $\{c_i\}$ and $\{d_i\}$, and the matrix elements $\langle i|f|j\rangle$.

c) Show that if $S = 0$ then this one-particle expectation value has the same result as if the particles were distinguishable and occupied two certain one-particle states (not necessarily basis states $|i\rangle$). Which are these certain states?

d) Now consider a two-particle operator V that is diagonal in the $|i\rangle$-basis (which means that $(ij|V|kl) = V_{ij} \, \delta_{ik} \, \delta_{jl}$). Calculate the expectation value of this operator in the state $|\Psi\rangle$ in terms of $\{c_i\}$, $\{d_i\}$ and V_{ij}. Show that the result is the same as for distinguishable particles, provided that $c_i \, d_i = 0$ for all i. Think about what this means when $\{|i\rangle\}$ is the position basis $\{|\mathbf{r}\rangle\}$.

2) Boson coherent states (optional): This question introduces a family of states known as boson coherent states. Consider a system with one eigenstate $\phi(\xi)$. As there is only one state, there is only one creation operator $a^\dagger$ and one annihilation operator a. Define the so-called coherent state $|z\rangle$, in which z is an arbitrary complex number, via

$$|z\rangle \equiv \exp(za^\dagger)|0\rangle.$$

a) Show that $|z\rangle$ is an eigenket of a with eigenvalue z.
b) Show that $\langle z'|z\rangle = \exp\left((z')^* z\right)$.

A string of creation and annihilation operators is said to be *normal-ordered operator* if all the creation operators occur to the left of all the annihilation operators. Given an un–normal-ordered operator, we construct its normal-ordered counterpart by re-ordering the string of creation and annihilation operators so that the annihilation operators all lie to the right of the creation operators. For any operator $f(a^\dagger, a)$, the normal-ordered counterpart is denoted $:f(a^\dagger, a):$.

c) Write down $:a\,a^\dagger:$. Show that for coherent states we have

$$\langle z'|\,:f(a^\dagger, a):\,|z\rangle = f\left((z')^*, z\right)\,\exp((z')^* z),$$

where $f\,(z', z)$ is any function that is analytic in both arguments.
d) Show that the identity operator may be written in the form

$$\mathrm{I} = \frac{1}{\pi}\int d\,\mathrm{Re}z\; d\,\mathrm{Im}z\; \mathrm{e}^{-z^* z}\,|z\rangle\langle z|.$$

e) Show that $\{|z\rangle\}$ is overcomplete.

Consider a single quantum-mechanical harmonic oscillator with coordinate q, momentum p, and hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 - \frac{1}{2}\hbar\omega.$$

f) At time $t = 0$ the oscillator is prepared in the coherent state $|z_{\mathrm{i}}\rangle$. Determine the state $|\Psi(t)\rangle$ at the subsequent time t. Does a coherent state remain coherent?
g) Compute the expectation values of q and p in the state $|\Psi(t)\rangle$.
h) Compare the expectation values obtained in part (i) with those obtained in energy eigenstates of the oscillator. Compare the time-dependence of the coherent-state expectation values of q and p with those of the classical oscillator.
i) Compute the uncertainties in q and p at time t. Comment on their product.
j) Compare the coherent state with the oscillator ground state.

3) Field operators for bosons: This question introduces so-called field operators for bosons. Recall that to define creation and annihilation operators we need a complete solution to a one-body eigenproblem, *i.e.*, a set of eigenfunctions $\psi_i(\xi)$, with eigenvalues ϵ_i. Then we

i) build the Hilbert space $\mathcal{I}_N$;
ii) symmetrise to obtain the boson space $\mathcal{B}_N$;
iii) collect together the boson spaces $\mathcal{B}_N$ and the vacuum space $\mathcal{B}_0$ to obtain the boson Fock space.

Creation and annihilation operators act on the occupation number representation of this Fock space in the following way:

$$\begin{aligned} a_j^\dagger|\ldots,n_j,\ldots\rangle &= \sqrt{n_j+1}\,|\ldots,n_j+1,\ldots\rangle, \\ a_j|\ldots,n_j,\ldots\rangle &= \sqrt{n_j}\,|\ldots,n_j-1,\ldots\rangle. \end{aligned}$$

Introduce the following linear combination of creation and annihilation operators, the so-called field operators:

$$\begin{aligned} \hat{\psi}^\dagger(\xi) &= \sum_j \psi_j^*(\xi)\,a_j^\dagger\,, \\ \hat{\psi}(\xi) &= \sum_j \psi_j(\xi)\,a_j\,. \end{aligned}$$

Think of these simply as sets of operators parametrised by ξ.

a) By recalling that $\psi_i(\xi)$ form a complete orthonormal set of functions for $\mathcal{I}_1$, prove that

$$[\hat{\psi}(\xi),\hat{\psi}^\dagger(\xi')] = \delta(\xi-\xi').$$

b) Show that for spinless noninteracting bosons in a potential $U(\mathbf{r})$ the hamiltonian may be written in terms of field operators in the following way:

$$H = -\frac{\hbar^2}{2m}\int d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r})\,\nabla^2\hat{\psi}(\mathbf{r}) + \int d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r})\,\hat{\psi}(\mathbf{r})\,U(\mathbf{r}).$$

c) Describe in physical terms the effect of applying $\hat{\psi}^\dagger(\mathbf{r})$ to a state. Determine is the physical quantity to which the operator $\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$ corresponds. Determine the physical quantity to which the operator $\int d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r})\,\hat{\psi}(\mathbf{r})$ corresponds.

d) Find the Heisenberg equation of motion for the operator $\hat{\psi}(\mathbf{r})$ by evaluating the relevant commutators for the hamiltonian of part (b).

e) Give a physical interpretation to the amplitude

$$\langle \mathrm{G}|\,\hat{\psi}(\mathbf{r}_2,\mathrm{t}_2)\,\hat{\psi}^\dagger(\mathbf{r}_1,\mathrm{t}_1)\,|\mathrm{G}\rangle,$$

where the time-dependence indicates that the operators are in the Heisenberg representation, $|\mathrm{G}\rangle$ is the ground state of the system, and you may assume that $t_2 > t_1$.

4) Nondegenerate perturbation theory: This question concerns first-order nondegenerate perturbation theory, couched in the language of the occupation number representation. Consider a system of identical particles (fermions or bosons) with hamiltonian

$$\mathcal{H} = \sum_i \epsilon_i \, a_i^\dagger \, a_i + \frac{1}{2} \sum_{qrst} a_q^\dagger \, a_r^\dagger \, a_s \, a_t \; (qr|V|ts) \, .$$

Show that the expectation value of $\mathcal{H}$ in the occupation number state $|n_1, n_2, \ldots\rangle$ is given by

$$E_{n_1,n_2,\ldots} = \sum_i \epsilon_i \, n_i + \frac{1}{2} \sum_{q \neq r} n_q \, n_r \, \{(qr|V|qr) \pm (qr|V|rq)\} + \frac{1}{2} \sum_q n_q \; (n_q - 1) \, (qq|V|qq) \, ,$$

where the $+(-)$ sign holds for bosons (fermions). The two matrix elements in the second term on the right hand side of this equation are known as the direct and exchange terms, respectively.