1) Noninteracting spin-3/2 fermions: Consider a system of N noninteracting spin-3/2 fermions in a cube of volume V with periodic boundary conditions.

a) Determine the single-particle energy spectrum.
b) Assuming that N and V are large, so that the sum may be replaced by an integral, compute the ground state energy as a function of $\hbar$, N, V and the particle mass m.
c) Compute the energy of the highest occupied single particle state (the so-called Fermi energy).

2) Boson Fock space: Consider the boson Fock space $\mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \cdots$. The creation operator is defined in the occupation number representation (ONR) by

$$a_j^\dagger|\ldots, n_j, \ldots\rangle = \sqrt{n_j+1}|\ldots, n_j+1, \ldots\rangle.$$

a) Evaluate the inner product $\langle n_1', \ldots | n_1, \ldots\rangle$.
b) Evaluate the matrix elements of $a_j^\dagger$ in the ONR.
c) By using the definition of $a_j^\dagger$ given in this question, show that $[a_j, a_k^\dagger] = \delta_{jk}$.
d) Consider noninteracting bosons. Suppose that the ONR is built up using the eigenkets of the one-body hamiltonian. Write down the total hamiltonian, in terms of creation and annihilation operators and the one-body eigenvalues.
e) Suppose that you are now given another one-body hamiltonian, the matrix elements of which in the basis of part (d) are h_{ij}. If this hamiltonian describes a collection of noninteracting bosons, write down the the total hamiltonian, in terms of creation and annihilation operators $a_j^\dagger$ and a_k and the matrix elements h_{ij}.
f) (**optional**) Prove that a general one-body operator $\sum_a f^{(a)}$ has the second quantised representation
$$\sum_{ij} a_i^\dagger\, a_j\, \langle i|f|j\rangle.$$
g) Consider free spinless bosons in a cube of volume V with periodic boundary conditions. A natural set of eigenfunctions is the set of normalised plane waves:
$$\psi_{\mathbf{p}}(\mathbf{r}) = \frac{1}{\sqrt{V}} \exp(i\mathbf{p}\cdot\mathbf{r}/\hbar).$$
Write down the total orbital angular momentum operator, in terms of creation and annihilation operators and a summation over momentum space.

h) Now consider noninteracting spin-half fermions in a cube of volume V with periodic boundary conditions. Write down the total spin operator, in terms of creation and annihilation operators, Pauli matrices and Planck's constant.

i) Write down the representation of a general two-body operator $\sum_{a<b} w^{(ab)}$ in second quantised form.

3) Anticommutation relations (optional): From the definition of fermion creation and annihilation operators given in class, *i.e.*, as operators that connect states of different particle number, establish the three anticommutation relations between the creation and annihilation operators.

4) Matrix representation of the creation and annihilation operators: Consider a particular single-particle state and a single species of fermion. Find a 2×2 matrix representation of the creation and annihilation operators. Show that your matrices satisfy the correct anticommutation relations (including having zero square).

5) Grand-canonical density operator: For a system consisting of a single species of identical particles the statistical mechanical density operator in the grand canonical ensemble is given by $\rho \equiv Z^{-1} \exp\left(-\beta\left(H - \mu N\right)\right)$, where $\beta \equiv 1/k_{\mathrm{B}}T$ (with T being the temperature), μ is the chemical potential, H is the Hamiltonian operator, N is the total particle-number operator, and the normalisation factor (*i.e.*, the partition function) $Z \equiv \mathrm{Tr}\exp\left(-\beta\left(H - \mu N\right)\right)$. The trace Tr is taken over a complete set of orthonormal physical states, *i.e.*, states with all possible occupation numbers (including all possible total particle numbers). Thermal averages are defined through

$$\langle A \rangle \equiv \mathrm{Tr} A\rho \equiv \sum_{n_1,\ldots} \langle n_1, \ldots | A\rho | n_1, \ldots \rangle.$$

a) If the particles are spin-1/2 fermions that are noninteracting and have eigenvalue spectrum $\epsilon_{\mathbf{k},\sigma}$, show that the mean number of particles in the state $(\mathbf{k}, \sigma)$ is given by the Fermi function:

$$\langle n_{\mathbf{k},\sigma} \rangle = \left\{1 + \exp\beta\left(\epsilon_{\mathbf{k},\sigma} - \mu\right)\right\}^{-1}.$$

b) If the particles are spinless bosons that are noninteracting and have eigenvalue spectrum $\gamma_{\mathbf{k}}$, show that the mean number of particles in the state $\mathbf{k}$ is given by the Bose function:

$$\langle n_{\mathbf{k}} \rangle = \left\{-1 + \exp\beta\left(\gamma_{\mathbf{k}} - \mu\right)\right\}^{-1}.$$

What restriction do you need to impose on μ?