1) Permutation operators (optional): The permutation operator $\hat{P}_A$ associated with the permutation

$$A \longleftrightarrow \begin{pmatrix} 1 & 2 & \cdots & N \\ a_1 & a_2 & \cdots & a_N \end{pmatrix}$$

acts in the following way: $\hat{P}_A|\mathbf{x}_1,\ldots,\mathbf{x}_N) = |\mathbf{x}_{\bar{a}_1},\ldots,\mathbf{x}_{\bar{a}_N})$, where

$$A^{-1} \longleftrightarrow \begin{pmatrix} 1 & 2 & \cdots & N \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_N \end{pmatrix}.$$

a) Prove that the permutation operator $\hat{P}$ is unitary.
b) Show that for a hamiltonian $\hat{H}$ of a system of identical particles, and a permutation operator $\hat{P}$, the following relation holds: $\hat{P}\hat{H}\hat{P}^{-1} = \hat{H}$.
c) Show that if $|\psi\rangle$ is a nondegenerate eigenket of a hamiltonian H for a system of identical particles, then $|\psi\rangle$ is either symmetric or antisymmetric under all pairwise exchanges.

2) Noninteracting particles: A single-particle quantum mechanical system possesses a Hilbert space spanned by three orthonormal eigenkets. Three particles occupy these states. How many distinct physical states are there if the three particles are:

a) Three identical fermions?
b) Three identical bosons?
c) Two identical fermions and a boson?
d) Two identical bosons and a fermion?
e) Three distinguishable fermions?
f) Three distinguishable bosons?

3) Identical spin-3/2 fermions: Consider 3 identical spin-3/2 fermions, for which spin and orbital degrees of freedom are not coupled. How many independent energy eigenstates are there associated with an *orbital* wave function $\psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ that is totally symmetric under permutation of its arguments?

4) Identical particles: Determine the conditions under which the following hamiltonians describe identical particles:

a) Two spin-1/2 degrees of freedom with

$$\hat{H} = \sum_{\mu,\nu=1}^{3} \hat{S}_{\mu}^{(1)} \, \Delta_{\mu\nu} \, \hat{S}_{\nu}^{(2)},$$

where $\Delta_{\mu\nu}$ are the arbitrary complex elements of a rank-2 tensor.

b) Three spin-1 bosons with

$$\hat{H} = \sum_{i=1}^{3} \frac{|\hat{\mathbf{p}}^{(i)}|^2}{2m^{(i)}} + \sum_{i=1}^{3} \gamma^{(i)} \hat{\mathbf{S}}^{(i)} \cdot \mathbf{B}^{(i)}(\hat{\mathbf{r}}^{(i)}) + \sum_{1 \leq i < j \leq 3} W^{(ij)}(\hat{\mathbf{r}}^{(i)}, \hat{\mathbf{r}}^{(j)}).$$

5) Exchange interaction: In this question we will examine the exchange interaction, introduced in class. Consider two electrons in an atom and neglect (i) spin-orbit coupling for each electron, and (ii) the electron-electron interaction. Suppose that the particles occupy the two orbitals $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$. Electrons are spin-1/2 particles: to each one we associate a spin observable $\hat{\mathbf{S}}$.

a) By including both spin and spatial degrees of freedom, build a basis of the possible physical states.

b) The electron-electron interaction $\hat{U}$ is spin-independent and translationally invariant. What does this tell us about its matrix elements?

c) By treating the electron-electron interaction to first order in perturbation theory, show that an effective hamiltonian for this set of states takes the form

$$\hat{H} = A\hat{\mathrm{I}} - \frac{J}{2}\left(\hat{\mathrm{I}} + \frac{4}{\hbar^2}\,\hat{\mathbf{S}}^{(1)} \cdot \hat{\mathbf{S}}^{(2)}\right).$$

What are A and J in terms of the orbital functions and the electron-electron interaction?

d) Now consider the four spin functions alone, and the action on them of the operator

$$\frac{1}{2}\left(1 + \frac{4}{\hbar^2}\,\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}\right).$$

Why is this operator called the exchange operator?

6) Real valued vectors and tensors: In this question we are going to consider the real-valued vectors and tensors with which you are familiar. However we shall use a notation which should help to illuminate the notation we have been using for Hilbert spaces in quantum mechanics.

Consider a d-dimensional linear vector space. Normally we would write an arbitrary vector as $\mathbf{t}$. It is a linear combination of unit vectors,

$$\mathbf{t} = \sum_{\mu=1}^{d} t_\mu \mathbf{e}_\mu,$$

where $\{\mathbf{e}_\mu\}$ is an orthonormal set of basis vectors and

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}.$$

Let us call this linear vector space G_1.

Simply change the notation:

$$\begin{aligned} \mathbf{t} &\to |t\rangle, && \text{an arbitrary vector;} \\ \mathbf{e}_\mu &\to |\mu\rangle, && \text{a basis vector;} \\ \mathbf{t}\cdot\mathbf{s} &= \langle t|s\rangle, && \text{an inner product.} \end{aligned}$$

Now consider the tensor

$$\sigma = \sum_{\mu\nu} \sigma_{\mu\nu}\, \mathbf{e}_\mu \otimes \mathbf{e}_\nu.$$

The set of tensors G_2 is spanned by the basis $\{\mathbf{e}_\mu \otimes \mathbf{e}_\nu\}$. Instead, we shall write

$$|\sigma\rangle = \sum_{\mu\nu} \sigma_{\mu\nu}\, |\mu\rangle \otimes |\nu\rangle = \sum_{\mu\nu} \sigma_{\mu\nu}\, |\mu,\nu\rangle.$$

Scalar products are defined by

$$(\mathbf{e}_\mu \otimes \mathbf{e}_\nu)\cdot(\mathbf{e}_\rho \otimes \mathbf{e}_\tau) = (\langle\mu| \otimes \langle\nu|)\,(|\rho\rangle \otimes |\tau\rangle) = \langle\mu|\rho\rangle\,\langle\nu|\tau\rangle = \delta_{\mu\rho}\,\delta_{\nu\tau}.$$

a) Evaluate $\langle\sigma|\omega\rangle$ in terms of the components $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$.
b) How many real numbers are required to parametrise elements of G_2?
c) Symmetric tensors are the elements of G_2 for which $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. How many real numbers are required to parametrise an arbitrary symmetric tensor?
d) How many elements are there in a basis for the symmetric tensors?
e) Write down a basis for the symmetric tensors for the case $d = 3$ in terms of the tensors $|\mu,\nu\rangle$.
f) Show that an arbitrary element of G_2 can be written as the sum of two pieces, one symmetric and one antisymmetric. Can this be done for an arbitrary element of G_3, where G_3 is defined as the obvious extension of G_1 and G_2?