Physics 581 | **Quantum Mechanics II** | P. M. Goldbart, 3135 ESB
Handout 11 | webusers.physics.illinois.edu/∼goldbart | University of Illinois
3 March 2010 | Homework 6 | goldbart@illinois.edu

1) Measurements and spin-1/2 particles:

a) A spin-1/2 particle is in a state with a definite value $\hat{S}_z = +\hbar/2$. Determine the probabilities of the possible outcomes of the measurement of $\mathbf{n} \cdot \hat{\mathbf{S}}$, where the unit vector $\mathbf{n}$ makes an angle θ with the z axis.

b) Reduce an arbitrary function of the scalar operator $a\hat{\mathrm{I}} + \mathbf{b} \cdot \hat{\mathbf{S}}$, linear in the spin-1/2 operator $\hat{\mathbf{S}}$, to another linear function.

2) Spin-1 algebra: Let $\hbar\hat{\boldsymbol{\sigma}}$ be the intrinsic angular momentum of a spin-1 particle, so that $\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\sigma}} = 2\hat{\mathrm{I}}$. Show that for any component $\hat{\sigma}_u \equiv \mathbf{u} \cdot \hat{\boldsymbol{\sigma}}$ (where $\mathbf{u}$ is any unit vector) we have:

a) $\hat{\sigma}_u^3 = \hat{\sigma}_u$;

b) $\exp\big(-i\varphi\hat{\sigma}_u\big) = 1 - i\hat{\sigma}_u \sin\varphi - \hat{\sigma}_u^2(1 - \cos\varphi)$.

3) Crown-shaped texture: Consider a neutral, spin-1/2 particle of mass m confined to a thin ring of radius a. Let the location of the particle on the ring be given by the plane polar angle θ. The spin of the particle is coupled an external static magnetic field $\mathbf{B}(\theta)$ via the Zeeman coupling:

$$H_{\mathrm{Z}} = -g\mathbf{B} \cdot \hat{\mathbf{S}}.$$

a) Write down the hamiltonian describing the system in terms of the z-component of the orbital angular momentum operator $\hat{L}_z$ and the coordinate operator $\hat{\theta}$.

Suppose that $\mathbf{B}(\theta)$ is given by the crown-shaped texture

$$\mathbf{B}(\theta) = \mathbf{e}_r \sin\chi + \mathbf{e}_z \cos\chi,$$

where $\{r, \theta, z\}$ are cylindrical polar coordinates and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the cylindrical polar basis vectors, and χ is a constant parameter.

b) Discuss the symmetry properties of the system, and the resulting conserved quantities, and list the good quantum numbers.

c) Determine the energy eigenvalue spectrum.

d) Compute the current in the ground state. When is your result nonzero? Is this consistent with the symmetry properties of the system?

4) Spin-1 freedom: Consider a quantum-mechanical system consisting of a single spin-1 degree of freedom. Let the state vectors $|\alpha\rangle$ (for $\alpha = 0, \pm 1$) be normalised eigenvectors of the z-component $\hat{S}_z$ of the angular momentum operator $\hat{\mathbf{S}}$, so that $\hat{S}_z|\alpha\rangle = \hbar\alpha|\alpha\rangle$ (for $\alpha = 0, \pm 1$). You may use without proof the following matrix representations of the operator $\hat{\mathbf{S}}$:

$$\langle\alpha|\hat{S}_x|\alpha'\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \langle\alpha|\hat{S}_y|\alpha'\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \langle\alpha|\hat{S}_z|\alpha'\rangle = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose that the hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = \hbar^{-1}\Omega(\mathbf{N}\cdot\hat{\mathbf{S}})^2,$$

where $\mathbf{N}$ is an arbitrary unit vector and Ω is an arbitrary positive constant (having the dimensions of an angular frequency).

a) Give all the possible outcomes of a measurement of the energy of this system.

b) Suppose that $\mathbf{N} = (\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ forms a cartesian basis. Find a complete orthonormal set of stationary state-vectors, expressing them in terms of $\{|\alpha\rangle\}$.

c) Suppose that $\mathbf{N} = \mathbf{e}_z$ and that the system is prepared at time $t = 0$ in the state $|\Psi\rangle$ given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |-1\rangle).$$

Calculate the amplitude that, at the subsequent time $t = T$, the system is again found to be in the state $|\Psi\rangle$.

d) Suppose that the system is prepared at a certain time in the state

$$|\Theta\rangle = \frac{1}{\sqrt{14}}(3|1\rangle + 2|0\rangle + |-1\rangle).$$

Immediately thereafter, the quantity $(\hat{S}_z)^2$ is measured, and the largest possible result is found. Give the normalised state vector immediately after the measurement.

e) Immediately after the measurement of $(\hat{S}_z)^2$ described in part (d) the quantity $\hat{S}_z$ is measured. Give the probability that the result $\hbar$ is obtained.

f) Suppose that $\mathbf{N} = \mathbf{e}_z$ but that Ω now varies with time as $\Omega(t) = \tilde{\Omega}\exp(-t/\tau)$, where $\tilde{\Omega}$ and τ are constants. Suppose further the system is again prepared at time $t = 0$ in the state $|\Psi\rangle$, given in part (c). Calculate the probability that, at time $t = \infty$, the system is found to be in the state $|\Psi\rangle$. State, giving a brief explanation, whether or not the z-component of angular momentum is conserved as the system evolves in time.