1) Super-adiabatic transitions (Landau-Zener tunnelling): Consider a quantum-mechanical system governed by a time-dependent hamiltonian that slowly evolves from H_{i} at time t_{i} to H_{f} at time t_{f}. If at t_{i} the system is in an eigenstate of H_{i} then at t_{f} the system will be in *the* eigenstate of H_{f} that derives from it by continuity, provided the evolution is infinitely slow (and certain other conditions hold: *e.g.*, continuity, differentiability, non-crossing of eigenvalues). This is the *quantum adiabatic theorem* of Born and Fock (see, *e.g.*, Messiah, *Quantum Mechanics* II, p. 739 *et seq.*).

The purpose of this question is to study the probability of *super-adiabatic* transitions, *i.e.*, quantum-number–changing transitions occurring in situations in which the evolution of the hamiltonian is slow but not infinitely so (*i.e.*, not quite adiabatic). The investigation of such processes is associated with the names of L. D. Landau and C. Zener; it is relevant, *e.g.*, to the physics of dielectric breakdown. We will study a model problem, due to Zener, which captures the essential physics.

Consider a two-state system having a Hilbert space spanned by the orthonormal pair of states $|1\rangle$ and $|2\rangle$. In this basis the hamiltonian is given by

$$\begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix} = \Delta \begin{pmatrix} t/T & 1 \\ 1 & -t/T \end{pmatrix},$$

where Δ and T are real positive parameters, and t is the time.

a) Find and sketch the instantaneous eigenvalues of H as a function of t. Contrast your results with those in the case $\Delta \to 0$ and $T \to 0$ with $\Delta/T = \delta$ (fixed).

Let $\epsilon^{\pm}(t)$ be the pair of (straight line) asymptotes to the instantaneous eigenvalues (such that $\epsilon^{+} < \epsilon^{-}$ for $t < 0$ and $\epsilon^{+} > \epsilon^{-}$ for $t > 0$). Express the general state in the form

$$|1\rangle\, C_1(t)\, \mathrm{e}^{-i\hbar^{-1}\int_0^t d\tau\, \epsilon^{+}(\tau)} + |2\rangle\, C_2(t)\, \mathrm{e}^{-i\hbar^{-1}\int_0^t d\tau\, \epsilon^{-}(\tau)},$$

where C_1 and C_2 are arbitrary time-dependent complex amplitudes.

b) By using the time-dependent Schrödinger equation construct a pair of coupled first-order ordinary differential equations satisfied by C_1 and C_2 [in terms of Δ and $\epsilon^{\pm}(\cdot)$].

At the initial time ($t \to -\infty$) the system is prepared in a state proportional to $|2\rangle$.

c) Give an initial condition for C_1 and C_2.
d) Give a ket proportional to the state of the system at all times, valid under the assumption that the evolution is truly adiabatic.

The evolution is *not* truly adiabatic, and our aim is to determine the (small) probability that, at late times, the system has undergone a quantum-number–changing transition.

e) Explain the connection between this probability and the amplitude $C_1(\infty)$.

f) Construct a second-order ordinary differential equation satisfied by C_1. By making suitable transformations of the dependent and independent variables, cast this equation into the standard form of the Weber equation, and hence show that the general solutions are parabolic cylinder functions. Note: You may wish to refer to one or more of the following books: *Abramowitz and Stegun* §19; *Gradshteyn and Ryzhik* §9.24-25; and *Whittaker and Watson's Modern Analysis* §16.5.

g) By considering the asymptotic properties of the parabolic cylinder functions, determine the asymptotic probability that the system suffers a quantum-number–changing transition. Identify an *adiabaticity parameter*. Briefly discuss the analytical behaviour of your result, as a function of the adiabatic parameter, in the adiabatic limit. Should your result be classified as *non-perturbative*?

Dykhne has derived an approximate formula for the probability P of a quantum-number–changing transition valid in the adiabatic limit:

$$P(- \to +) \sim \mathrm{e}^{-2\hbar^{-1}\,\mathrm{Im}\int_0^{t_c} dt\,\{E^+(t)-E^-(t)\}},$$

where $E^+(t)$ and $E^-(t)$ [with $E^+(t) > E^-(t)$] are instantaneous eigenvalues of the (non-degenerate) hamiltonian, and t_c is the (complex) root of the degeneracy condition $E^+(t) = E^-(t)$. It is easier to obtain results by using Dykhne's formula than by following Zener's asymptotic analysis.

h) Obtain the probability of a quantum-number–changing transition for the Zener problem by applying Dykhne's formula. Compare your result with that obtained in part (g).

In contrast with their *probabilities*, the quantal *phases* of adiabatically-evolving quantum states were not thoroughly studied until surprisingly recently. In 1984, Michael Berry published a beautiful analysis of this issue [M. V. Berry, *Proc. R. Soc. Lond.* A **392** (1984) 45-57]. This has richly elucidated the subject by identifying an important and unifying concept known as the geometric (or Berry) phase.

2) Spin-1/2 dynamics (optional): Consider the quantum dynamics of a spin-1/2 particle.

a) $\hat{S}_y$ is measured at time $t = 0$ and the result $\hbar/2$ is found. Give a state vector $|\psi(t = 0)\rangle$ immediately after the measurement.

Between $t = 0$ and $t = T$ a uniform magnetic field is applied. This field depends on time, and the hamiltonian becomes $\hat{H} = \Omega_0(t)\hat{S}_z$.

b) Let $\Omega_0(t) = \bar{\Omega}t/T$, where $\bar{\Omega}$ is a constant frequency. Show that for times $0 \leq t \leq T$ the state vector takes the form

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left\{ \mathrm{e}^{i\theta(t)}|+\rangle + i\mathrm{e}^{-i\theta(t)}|-\rangle \right\},$$

and find the appropriate function $\theta(t)$.

c) The field is removed at time $t = T$, and at some later time $\hat{S}_y$ is measured. Calculate the probability of finding the result $\hbar/2$.

d) State the values of T for which one can be certain of the outcome of the measurement of $\hat{S}_y$.