1) Nuclear magnetic resonance: The cartesian components of the spin-1/2 Pauli operator $\hat{\boldsymbol{\sigma}}$ obey $\hat{\sigma}_j\,\hat{\sigma}_k = \delta_{jk}\,\hat{\mathrm{I}} + i\epsilon_{jkm}\,\hat{\sigma}_m$, where $\hat{\mathrm{I}}$ is the the identity operator. The spin-1/2 Hilbert space rotation operator corresponding to a rotation through an angle θ about an axis $\boldsymbol{\theta}/\theta$, is given by $\hat{\mathcal{R}}(\boldsymbol{\theta}) = \exp\left(\boldsymbol{\theta}\cdot\hat{\boldsymbol{\sigma}}/2i\right)$. The spin is governed by the hamiltonian $\hat{\mathcal{H}} = -(\hbar\boldsymbol{\Omega}/2)\cdot\hat{\boldsymbol{\sigma}}$.

a) Derive the Heisenberg equation of motion for the Heisenberg-picture operator $\hat{\boldsymbol{\sigma}}(t)$. Briefly describe the form of motion that the expectation value of $\hat{\boldsymbol{\sigma}}(t)$ exhibits.
b) Demonstrate that the ket $|\psi\rangle_t \equiv \hat{\mathcal{R}}(-\boldsymbol{\Omega}t)\,|\psi\rangle_0$ satisfies the appropriate time-dependent Schrödinger equation.
c) Construct an explicit expression for $\hat{\mathcal{R}}(\boldsymbol{\theta})$, linear in both $\hat{\boldsymbol{\sigma}}$ and $\hat{\mathrm{I}}$.

Now suppose that $\boldsymbol{\Omega}$ is aligned along the z axis and that the spin is subject to an additional, time-dependent torque, so that its hamiltonian becomes

$$\hat{\mathcal{H}} = -(\hbar\Omega/2)\left(\hat{\sigma}_z + \lambda\,\hat{\sigma}_x\,\cos\omega t - \lambda\,\hat{\sigma}_y\,\sin\omega t\right),$$

where Ω and ω are positive frequencies and λ is a positive constant.

d) Give a simple sketch to indicate the properties of the field acting on the spin.
e) Show that the ket

$$|\psi\rangle_t = \hat{\mathcal{R}}(-\omega t\mathbf{e}_z)\,\hat{\mathcal{R}}(-\boldsymbol{\Gamma}t)\,|\psi\rangle_0\,,$$

in which $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ form an orthonormal cartesian basis, satisfies the relevant time-dependent Schrödinger equation. State the necessary value of $\boldsymbol{\Gamma}$ in terms of λ, Ω, ω. *Hint*: You may find it helpful to transform to a basis rotating about the z axis at a frequency ω, *i.e.*, to exchange the equation of motion obeyed by the Schrödinger picture state vector $|\psi\rangle_t$ for an equation of motion obeyed by a state vector

$$|\widetilde{\psi}\rangle_t \equiv \hat{\mathcal{R}}(\omega t\mathbf{e}_z)\,|\psi\rangle_t\,.$$

Be sure to *think* about the effects of $\hat{\mathcal{R}}(\pm\omega t\mathbf{e}_z)$, especially on operators, rather than just launch into some extended calculation.
f) Compute the *exact* probability for the system to be in the spin-down state (with respect to the z-axis) at time t (> 0), given that is was in the spin-up state at time $t = 0$. Express your answer in terms of λ, Ω, ω and t.
g) Identify a circumstance under which an approach based on time-dependent perturbation theory would be *invalid*, even if the parameter λ were numerically small. Briefly explain the physical origin of this phenomenon.

2) Spin-1/2 Hilbert space: Consider the spin-1/2 Hilbert space, in which the kets $|\pm\rangle$ are eigenkets of $\hat{S}_z$ with eigenvalues $\pm\hbar/2$. Define the unit vector $\mathbf{n}$ in terms of the spherical polar coordinates $\{\theta, \phi\}$: $\mathbf{n} = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z$.

a) The ket $|\mathbf{n}, +\rangle$ is defined by $|\mathbf{n}, +\rangle \equiv \cos(\theta/2)\mathrm{e}^{-i\phi/2}|+\rangle + \sin(\theta/2)\mathrm{e}^{i\phi/2}|-\rangle$. Show that it is an eigenket of the operator $\mathbf{n}\cdot\hat{\mathbf{S}}$. Compute the corresponding eigenvalue? Write down a ket that has eigenvalue $-\hbar/2$?

b) Given a state vector $|\psi\rangle$ one can, of course, compute the expectation value of the spin operator $\langle\psi|\hat{\mathbf{S}}|\psi\rangle$. Show that the converse is true for spin-1/2, *i.e.*, that you can work *backwards* from the expectation value to obtain $|\psi\rangle$ (to within a phase factor).

c) Explain why every ket in spin-1/2 Hilbert space is an eigenket, with eigenvalue $\hbar/2$, of the operator $\mathbf{n}\cdot\hat{\mathbf{S}}$ for *some* unit vector $\mathbf{n}$.

d) Explain why, and also verify by explicit calculation that, $|\mathbf{n}, +\rangle = \hat{R}(\phi\mathbf{e}_z)\,\hat{R}(\theta\mathbf{e}_y)\,|+\rangle$.

3) Spin-operator algebra – optional: Let $\mathbf{A}$ and $\mathbf{B}$ be classical vectors (or operators that commute with $\boldsymbol{\sigma}$). By using the properties of the spin operators $\boldsymbol{\sigma}$ show that

$$(\mathbf{A}\cdot\boldsymbol{\sigma})\,(\mathbf{B}\cdot\boldsymbol{\sigma}) = (\mathbf{A}\cdot\mathbf{B})\,\hat{\mathrm{I}} + i\,(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma},$$

where $\hat{\mathrm{I}}$ is the identity operator.

4) More spin-operator algebra: Consider any complex 2×2 matrix $\mathcal{M}$.

a) Show that $\mathcal{M}$ can always be written in the form $\mathcal{M} = \sum_{i=0}^{3}\mu_i\sigma_i$, where $\{\sigma_i\}_{i=0}^{3}$ are the Pauli *matrices* (including the identity matrix I for $i = 0$).

b) Show that the expansion coefficients $\{\mu_i\}_{i=0}^{3}$ may be extracted from $\mathcal{M}$ as

$$\mu_i = (1/2)\mathrm{Tr}\,(\mathcal{M}\sigma_i).$$

Express the following quantities as linear combinations of the Pauli matrices and the identity matrix:

c) $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, d) $(\mathrm{I} + i\sigma_1)^{1/2}$, e) $(2\mathrm{I} + \sigma_1)^{-1}$, and f) $(\sigma_1)^{-1}$.

5) Yet more spin-operator algebra – optional:

a) Show that any matrix that commutes with all three Pauli matrices must be a multiple of the identity matrix.

b) Show that any matrix that anticommutes with all three Pauli matrices must be the zero matrix.