1) Incoherence: The purpose of this question is to highlight the ideas behind incoherence. Suppose a certain transition rate Γ is determined by a sum of complex variables $\{z_n\}_{n=1}^N$ in the following way: $\Gamma = |\sum_{n=1}^N z_n|^2$. Now suppose that whilst the amplitudes $\{r_n\}_{n=1}^N$ of $\{z_n\}_{n=1}^N$ are fixed, the phases $\{\phi_n\}_{n=1}^N$ are independent identically distributed random variables chosen from a uniform distribution on the interval $0 \le \phi_n < 2\pi$. Show that the mean and variance of Γ are given by

$$[\Gamma] = \sum_{n=1}^N r_n^2 \qquad \text{and} \qquad [\Gamma^2] - [\Gamma]^2 = \left(\sum_{n=1}^N r_n^2\right)^2 - \sum_{n=1}^N r_n^4,$$

where the brackets $[\cdots]$ denote averaging over the random phase variables.

2) Incoherent perturbations:

a) State Fermi's golden rule for the case of the time-dependent perturbation $\hat{H}_1(t) = \hat{U}\cos\omega t$, stating carefully the meaning of the symbols you write. List the conditions under which this rule is valid.

An unperturbed system consists of a particle of mass m confined to a large three-dimensional cubic region of volume V on which periodic boundary conditions are imposed. A perturbation is applied that is a superposition of perturbations of closely spaced frequencies ω_n and strengths g_n. The superposition is *incoherent*; the perturbation at frequency ω_n is

$$g_n \cos(\omega_n t - \phi_n) \cos(\mathbf{Q}\cdot\hat{\mathbf{r}}),$$

where $\mathbf{Q}$ is a fixed, nonzero wave vector, and ϕ_n (which vary over the ensemble) specify the random phase associated with each frequency.

b) How does the incoherence of the perturbation allow you to simplify the calculation of the scattering rate between two states?

c) Calculate the total scattering rate out of the ground state due to the incoherent perturbation, expressing your answer in terms of the spectral density $\gamma(\omega) \equiv \sum_n g_n^2\,\delta(\omega - \omega_n)$, for the perturbation.

3) Scattering rates: A particle of mass m is confined to a large two-dimensional square of area A, where it experiences a weak external potential $V(\mathbf{r})$.

a) Use Fermi's golden rule, or any other *time-dependent* method, to compute the scattering rate from an incoming momentum eigenstate $|\hbar\mathbf{k}\rangle$ into a sector of angle $d\theta$ around the direction $\hat{\mathbf{l}}$.

b) By relating the scattering rate to the incoming flux, find the differential scattering cross-section for the target described by the potential $V(\mathbf{r})$. Express your answer in terms of the Fourier transform of the potential.

c) Now consider an *incoherent* incident beam that consists of particles with momenta $n\hbar\mathbf{q}$, where $n = 0, \pm 1 \pm 2, \ldots$, *i.e.*, particles with a spectrum of momenta incident from both directions along the same axis. Suppose that momentum $n\hbar\mathbf{q}$ occurs with probability p_n. Calculate, in terms of p_n, the scattering rate into a sector of angle $d\theta$ around the direction $\hat{\mathbf{l}}$ (with $\mathbf{l} \neq n\mathbf{q}$).

d) Suppose that the potential $V(\mathbf{r})$ and the probability distribution p_n are gaussian, *i.e.*,

$$V(\mathbf{r}) = g\exp\left(-|\mathbf{r}|^2/2a^2\right) \qquad \text{and} \qquad p_n \propto \exp\left(-n^2/2N^2\right),$$

and that the sum $\sum_n$ may be replaced by the integral $\int dn$. Calculate the scattering rate into a sector of angle $d\theta$ around the direction $\hat{\mathbf{l}}$ (with $\mathbf{l} \neq n\mathbf{q}$).

4) Gauge transformations – optional:

a) Write down the time-dependent Schrödinger equation for the wave function of a particle of charge q in the presence of both a scalar potential $\phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$.

Under a gauge transformation

$$\begin{aligned} \phi(\mathbf{r}, t) \to \phi'(\mathbf{r}, t) &= \phi(\mathbf{r}, t) + \partial\Lambda(\mathbf{r}, t)/\partial(ct), \\ \mathbf{A}(\mathbf{r}, t) \to \mathbf{A}'(\mathbf{r}, t) &= \mathbf{A}(\mathbf{r}, t) - \nabla\Lambda(\mathbf{r}, t). \end{aligned}$$

b) Write down, and demonstrate the correctness of, the transformation rule for the wave function $\psi(\mathbf{r}, t)$ so that if $\psi(\mathbf{r}, t)$ satisfies the Schrödinger equation in the presence of $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ then $\psi'(\mathbf{r}, t)$ satisfies the same equation but with $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ replaced by $\phi'(\mathbf{r}, t)$ and $\mathbf{A}'(\mathbf{r}, t)$.

c) Give an example of an operator whose matrix elements are gauge-invariant. Show that its matrix elements are indeed independent of the gauge.

5) Absorption of light by a model atom – optional: Calculate the absorption rate due to the ejection of electrons from a model atom stimulated by a weak incoherent classical radiation field $\mathbf{A}(\mathbf{r},t)$ given by

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}p} \left(A_{\mathbf{k}p}\, \mathbf{l}_p \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r} - ickt} + \text{c. c.} \right).$$

You may use the following information without proof:

(i) The initial state of the electron is the ground state of a three-dimensional harmonic oscillator, *i.e.*, $\langle \mathbf{r}|\psi_i\rangle = (\pi a^2)^{-3/4} \exp\left(-r^2/2a^2\right)$, and its energy is $\epsilon_i < 0$.

(ii) The final state wave function of the electron is a particular plane wave with momentum $\hbar\mathbf{q}$.

(iii) The interaction $\mathcal{H}_{\text{int}}(t)$ between the electron and the radiation field is given by

$$\mathcal{H}_{\text{int}}(t) = -\frac{q}{c} \frac{1}{\sqrt{V}} \sum_{\mathbf{k}p} \left(A_{\mathbf{k}p}\, \mathbf{l}_p \cdot \mathbf{j}_{-\mathbf{k}}\, \mathrm{e}^{-ickt} + \text{h. c.} \right).$$

(iv) The current operator is given by $\mathbf{j}_\mathbf{k} = \left(\mathbf{p}\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} + \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}\, \mathbf{p} \right)/2m$.

(v) Fermi's golden rule is valid, although you should state it clearly.

(vi) $\int d^3x \, \exp\left(-x^2/2 + i\mathbf{m}\cdot\mathbf{x}\right) = (2\pi)^{3/2} \exp\left(-m^2/2\right)$.

6) Maxwell's equations – optional: Begin with Maxwell's equations for the electromagnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ in a cube of volume V with periodic boundary conditions in the absence of charges and currents. Demonstrate that in the Coulomb gauge the vector potential $\mathbf{A}(\mathbf{r},t)$ may be entirely parametrised by the two complex-valued fields $A_{\mathbf{k},1}$ and $A_{\mathbf{k},2}$, in the sense that

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sum_{p=1}^{2} \left\{ A_{\mathbf{k},p}\, \mathbf{l}_p\, e^{i\mathbf{k}\cdot\mathbf{r} - ickt} + A^*_{\mathbf{k},p}\, \mathbf{l}_p\, e^{-i\mathbf{k}\cdot\mathbf{r} + ickt} \right\},$$

where $\mathbf{l}_1$ and $\mathbf{l}_2$ are polarisation vectors, $\{\mathbf{l}_1, \mathbf{l}_2, \hat{\mathbf{k}}\}$ forms an orthonormal triad of real vectors for each $\mathbf{k}$, and $\hat{\mathbf{k}}$ denotes the unit vector parallel to $\mathbf{k}$.

7) Homogeneous magnetic fields – optional: A vector potential $\mathbf{A}(\mathbf{r})$ is written in terms of a constant vector $\mathbf{h}$ in the following way: $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{h} \times \mathbf{r}$. Compute the corresponding magnetic field $\mathbf{B}$.

8) Vector potentials – optional: Consider the vector potential due to a single Fourier component with wave vector $\mathbf{q}$. Suppose that $A_{\mathbf{q},1} = 1$ and $A_{\mathbf{q},2} = e^{i\phi}$. Compute $\mathbf{A}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$. Discuss the polarisation of $\mathbf{A}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ for the cases: (i) $\phi = 0$; and (ii) $\phi = \pi/2$.

9) Stimulated emission – optional: Stimulated emission is the process of emission of radiation in the presence of a radiation field. By following a line of argument similar to that used in class for the investigation of the rate of *absorption* of incoherent radiation by atoms, calculate the *emission* rate stimulated by incoherent radiation. Show that this rate is equal to the absorption rate found in class.