1) Conservation of energy: A system is described by a time-independent hamiltonian $H = H_0 + H_1$. At time $t = 0$ the system is found in the state $|a\rangle$, which is an eigenket of H_0 with eigenvalue ϵ_a. $P_{a\to b}(t)$ is defined to be the probability of finding the system in state $|b\rangle$, another eigenket of H_0, at time t. $|a\rangle$ and $|b\rangle$ are orthonormal.

a) **(optional)** By starting with the time dependent Schrödinger equation, show that to first order in the perturbation we have

$$P_{a\to b}(t) = \frac{t^2}{\hbar^2} \left|\langle b|H_1|a\rangle\right|^2 f(\omega_{ba}\, t),$$

where $\hbar\omega_{ba} \equiv \epsilon_b - \epsilon_a$.

b) **(optional)** Determine the function $f(x)$?

c) **(optional)** Define the transition rate $W_{a\to b}$ via $W_{a\to b} \equiv \lim_{t\to\infty} t^{-1} P_{a\to b}(t)$. By using the result $\lim_{x\to\infty} (y^2 x/4)^{-1} \sin^2(yx/2) = 2\pi\, \delta(y)$, or otherwise, find $W_{a\to b}$.

d) **(optional)** Is there anything that you can say about the unperturbed energies, ϵ_a and ϵ_b?

e) What does the result for $W_{a\to b}$ become when H_1 is replaced by the time-dependent perturbation $H_2(t)$, given by

$$H_2(t) = S\cos(\sigma t) + T\cos(\tau t)?$$

f) Suppose that both first- and second-order transitions are allowed for the hamiltonian of part (e). State the possible values of the unperturbed energy of the final state.

2) Fermi's golden rule for oscillatory perturbations:

a) State Fermi's golden rule for the oscillatory perturbation

$$H_1(t) = U\cos(\omega t).$$

b) A spinless particle of mass m moves in a large four-dimensional hypercube of volume V with periodic boundary conditions. It experiences the "weak" oscillatory perturbing potential,

$$\frac{\hbar^2\lambda}{2m}\,\delta(\mathbf{r})\,\cos(\omega t),$$

acting at the centre of the box. Calculate, to first order, the total scattering rate out of the ground state. (You may use without proof the result that the surface *volume* of a four-dimensional sphere of unit radius is $2\pi^2$.)

3) Ionisation of atomic hydrogen: A hydrogen atom consists of an electron of mass m and a proton of mass $m_\mathrm{p} \gg m$ interacting via a Coulomb potential $-e^2/r$. The atom is subject to an oscillating electric field $\mathbf{E}\cos\omega t$, and is initially in the ground state. By assuming that the eigenfunctions of the continuous part of the spectrum are adequately described by plane waves, and that the electric field is weak, compute the ionisation rate as a function of the frequency ω. Establish the angular distribution of the emitted electrons.

4) Coulomb excitation: A hydrogen atom is fixed at the origin of a coordinate system. A heavy particle, of charge Ze, which we shall treat classically, is projected with speed V along the trajectory

$$\mathbf{R}(t) = Vt\mathbf{e}_x + D\mathbf{e}_y,$$

so that its point of closest approach to the nucleus of the hydrogen atom occurs at time $t = 0$, when the separation is D.

a) Write down the operator $H_1(t)$ that describes the Coulomb interaction between the heavy particle and the electron (which orbits the hydrogen atom and has charge $-e$) in terms of Z, e, $\mathbf{R}(t)$, and the electron position operator $\mathbf{r}$.

b) Consider the classical particle as providing a time-dependent perturbation of the hydrogen atom. Derive a formula for the first-order probability, $P_{n\to m}$, that if the electron is initially in the atomic eigenstate $|n\rangle$ with energy eigenvalue ϵ_n then it will end up in an orthogonal eigenstate $|m\rangle$ with energy eigenvalue ϵ_m. At this stage, do not attempt to evaluate any integrals.

c) By assuming that D^2 is large compared with the mean square radius of the electron orbitals $|n\rangle$ and $|m\rangle$, show that to leading order,

$$P_{n\to m} \simeq \frac{Z^2e^4}{\hbar^2}\Big|\int_{-\infty}^{\infty} dt\, \frac{(Vtx_{mn} + Dy_{mn})}{(V^2t^2 + D^2)^{3/2}}\, \mathrm{e}^{i\omega_{mn}t}\Big|^2,$$

where $\hbar\omega_{mn} \equiv \epsilon_m - \epsilon_n$, $x_{mn} \equiv \langle m|x|n\rangle$, $y_{mn} \equiv \langle m|y|n\rangle$, and x and y are components of the electron position operator $\mathbf{r}$.

d) What range of values of t give the dominant contribution to the integral in part (c)? Refer to this scale of times as τ.

e) Now suppose that $\omega_{mn}\tau \ll 1$. By making use of the substitution $Vt = D\tan\theta$, evaluate the integral, and hence find $P_{n\to m}$.