1) Harmonic oscillator: Consider the one-dimensional oscillator

$$H_0 = \frac{1}{2}(p^2 + x^2 - 1).$$

(To ease notation we adopt units in which $\hbar = 1$, the mass $m = 1$ and the natural frequency $\omega = 1$.) Introduce the raising and lowering operators $a^\dagger$ and a, respectively defined by

$$a \equiv \frac{1}{\sqrt{2}}(x + ip) \qquad \text{and} \qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip).$$

a) Express H_0 in terms of a and $a^\dagger$.
b) Compute the matrix elements $\langle n|a^\dagger|m\rangle$, where $|n\rangle$ is the eigenket of H_0 with eigenvalue $n = 0, 1, \ldots$.

Consider the perturbation $H_1(t)$ given by

$$H_1(t) = gx\mathrm{e}^{-t^2/2\tau^2},$$

i.e., a time-dependent external force. Suppose that at time $t_0 \ll -\tau$ the system is in the state $|n\rangle$. Use time-dependent perturbation theory to find:

c) the amplitude, to $(m-1)^{\text{th}}$ order in perturbation theory, for finding the system in the state $|n + m\rangle$ at time $t \gg \tau$;
d) this amplitude to m^{th} order; and
e) this amplitude to $(m + 1)^{\text{th}}$ order.
f) Discuss the number of terms that you expect to contribute to the amplitude at $(m+2)^{\text{th}}$ order.

Now consider the perturbation $H_1(t)$ given by

$$H_1(t) = \beta x^2\mathrm{e}^{-t^2/2\tau^2}.$$

g) Give a physical interpretation of this perturbation.
h) At time $t_0 \ll -\tau$ the system is in the state $|n\rangle$. Compute the first-order probability for finding the system in the state $|m\rangle$ at time $t \gg \tau$, assuming that $m \neq n$.

2) Two-level system: A system is described by a Hilbert space spanned by two orthonormal kets $|1\rangle$ and $|2\rangle$. In this basis the matrix elements of the hamiltonian H_0 are

$$\begin{pmatrix} \langle 1|H_0|1\rangle & \langle 1|H_0|2\rangle \\ \langle 2|H_0|1\rangle & \langle 2|H_0|2\rangle \end{pmatrix} = \begin{pmatrix} 2\Lambda\hbar & 0 \\ 0 & 0 \end{pmatrix},$$

where Λ is real. At time $t = 0$ the system is in state $|1\rangle$ and a perturbation, H_1, the matrix elements of which are

$$\begin{pmatrix} \langle 1|H_1|1\rangle & \langle 1|H_1|2\rangle \\ \langle 2|H_1|1\rangle & \langle 2|H_1|2\rangle \end{pmatrix} = \begin{pmatrix} 0 & \lambda\hbar \\ \lambda\hbar & 0 \end{pmatrix}$$

(with λ real), is sharply switched on.

a) Show that the eigenvalues and eigenkets of $H_0 + H_1$ are respectively given by $E_\pm = \hbar(\Lambda \pm \Delta)$, and

$$\begin{pmatrix} \langle 1|\mu_+\rangle \\ \langle 2|\mu_+\rangle \end{pmatrix} = d \begin{pmatrix} \Lambda + \Delta \\ \lambda \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \langle 1|\mu_-\rangle \\ \langle 2|\mu_-\rangle \end{pmatrix} = d \begin{pmatrix} -\lambda \\ \Lambda + \Delta \end{pmatrix},$$

where $\Delta^2 \equiv \Lambda^2 + \lambda^2$ and $d^{-2} \equiv 2\Delta(\Lambda + \Delta)$.

b) Show that the probability of finding the system in state $|2\rangle$ at time t, given that it was in state $|1\rangle$ at time 0, is given by $(\lambda^2/\Delta^2)\sin^2 \Delta t$.

c) By using time-dependent perturbation theory to first order, find an approximate expression for the probability in part (b).

d) By expanding the exact amplitude of part (b), recover the perturbative result of part (c).

3) Brief perturbation: Consider a system subject to the perturbation $H_1(t) = U\delta(t)$. At time $t = 0^-$ the system is in the state $|i\rangle$. Compute the first-order amplitude for finding the system in the state $|f\rangle$, orthogonal to $|i\rangle$, at time $t = 0^+$? Note that, although the perturbation is infinite, we can still use perturbation theory provided that "the area under it" is sufficiently small.

4) Lagrangian for a charged particle – optional: The lagrangian $\mathcal{L}$ for a particle of charge q and mass m in the presence of the external electromagnetic potentials $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ is given by

$$\mathcal{L} = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) - q\phi(\mathbf{r}, t).$$

a) By starting with this lagrangean, derive the Lorentz equation of motion.

b) Construct the canonical momentum.

c) Derive the hamiltonian.

d) Show that under a gauge transformation the lagrangean changes by a total derivative. What effect does this have on the classical equations of motion?

e) Read *Baym*, pp. 74-79.