Physics 581 **Quantum Mechanics II** P. M. Goldbart, 3135 ESB
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This particular homework is **optional**, but I encourage you to work through the problems.

**1) Adiabatic approximation and Berry's geometric phase**: In this question we shall consider the dynamics of a quantum-mechanical spin-1/2 degree of freedom $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$ governed by the Hamiltonian

$$H(t) = -B\,\mathbf{n}(t)\cdot\boldsymbol{\sigma}\,.$$

The time-dependent unit vector $\mathbf{n}(t)$ describes the instantaneous orientation of the magnetic field, and the constant $B$ describes its magnitude. We introduce the pair of states $|\pm,\mathbf{e}_z\rangle$ which are eigenstates of $\sigma_z$, *i.e.*, $\sigma_z|\pm,\mathbf{e}_z\rangle = \pm|\pm,\mathbf{e}_z\rangle$, where $\mathbf{e}_z$ is one of three orthonormal basis vectors $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$. Furthermore, it will be convenient to parametrise $\mathbf{n}$ in terms of spherical polar coordinates, *i.e.*, $\{n_x,n_y,n_z\} = \{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\}$

a) Consider the states $|+,\mathbf{n}\rangle$ defined by

$$\begin{aligned} |+,\mathbf{n}\rangle &\equiv +\mathrm{e}^{-i\phi/2}\cos(\theta/2)|+,\mathbf{e}_z\rangle + \mathrm{e}^{+i\phi/2}\sin(\theta/2)|-,\mathbf{e}_z\rangle, \\ |-,\mathbf{n}\rangle &\equiv -\mathrm{e}^{-i\phi/2}\sin(\theta/2)|+,\mathbf{e}_z\rangle + \mathrm{e}^{+i\phi/2}\cos(\theta/2)|-,\mathbf{e}_z\rangle. \end{aligned}$$

Show that the states $|\pm,\mathbf{n}\rangle$ are eigenstates of the operator $\mathbf{n}\cdot\boldsymbol{\sigma}$.

a-i) Calculate the corresponding eigenvalues?

a-ii) Is there degeneracy for any value of $\mathbf{n}$?

a-iii) Are the states $|\pm,\mathbf{n}\rangle$ single-valued functions of $\mathbf{n}$?

We refer to the states $|\pm,\mathbf{n}\rangle$ as instantaneous energy eigenstates because

$$H(t)|\pm,\mathbf{n}(t)\rangle = E_\pm(\mathbf{n}(t))|\pm,\mathbf{n}(t)\rangle,$$

where $E_\pm(\mathbf{n}(t)) = \mp B$, and we refer to $E_\pm(n(t))$ as instantaneous energy eigenvalues.

Now, according to the adiabatic theorem, provided (i) that the magnetic field $B\mathbf{n}(t)$ varies sufficiently slowly, and (ii) that no degeneracy is encountered, a system prepared at time $t = 0$ in a certain instantaneous eigenstate of $H(0)$ will evolve into the eigenstate of $H(t)$ with the same quantum number, *i.e.*, if $|\psi(0)\rangle = |+,\mathbf{n}(0)\rangle$ then

$$|\psi(t)\rangle = |+,\mathbf{n}(t)\rangle \exp\Big(i\gamma_+(t) - \frac{i}{\hbar}\int_0^t dt' E_+(\mathbf{n}(t'))\Big),$$

where $\gamma_+(t)$ is some phase angle, and $\gamma_+(0)$ may be chosen to vanish.

b) By requiring that $|\psi(t)\rangle$ approximately satisfies the time-dependent Schrödinger equation, find an expression for $d\gamma_+(t)/dt$ in terms of the inner product $\langle +,\mathbf{n}|\Big\{\frac{\partial}{\partial\mathbf{n}}|+,\mathbf{n}\rangle\Big\}$, evaluated at $\mathbf{n} = \mathbf{n}(t)$, and the rate of change of the orientation of the magnetic field $d\mathbf{n}(t)/dt$.

c) Now suppose that $\mathbf{n}(t)$ executes a closed path $\mathcal{C}$, so that $\mathbf{n}(T) = \mathbf{n}(0)$. By integrating your answer to part (b), or otherwise, show that

$$\gamma_+(T) = i \oint_{\mathcal{C}} \langle +, \mathbf{n}| \Big\{ \frac{\partial}{\partial \mathbf{n}} |+, \mathbf{n}\rangle \Big\} \cdot d\mathbf{n},$$

and hence show that the resulting phase $\gamma_+(T)$ depends only on the geometry of the closed path $\mathcal{C}$ and not on the manner in which it is traversed.

In fact, it can be shown that $\gamma_+(T)$ is simply related to one half of the solid angle swept out by $\mathbf{n}$ as it completes its closed path closed path $\mathcal{C}$, a rather appealing result due to M. V. Berry. For a discussion of this and many related issues, see A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1989), especially the article by M. V. Berry [*Proc. R. Soc. Lon.*, Ser. A **392**, 45 (1984)] reprinted therein.

**2) A version of Foucault's pendulum**: Consider a classical point particle of mass $m$ attached to the origin $\mathcal{O}$ of three-dimensional space by the harmonic potential $U(\mathbf{r}) = m\omega^2|\mathbf{r}|^2/2$, where $\mathbf{r}$ is the position vector of the particle and the frequency $\omega$ is a (real) parameter. The particle is constrained to a plane that contains $\mathcal{O}$, the unit vector normal to the plane being $\mathbf{N}$. Suppose that for times $|t| > T/2$ the normal $\mathbf{N}$ does not vary, but that for times $|t| < T/2$ it does vary, albeit adiabatically slowly (*i.e.*, $|\dot{\mathbf{N}}| \ll \omega$). Suppose, further, that for $t < -T/2$ the particle oscillates linearly, along a direction $\mathbf{S}$.

a) Show that it the adiabatic limit the oscillation direction obeys the equation of motion

$$\frac{d\mathbf{S}}{dt} = \left(\mathbf{N} \times \dot{\mathbf{N}}\right) \times \mathbf{S}.$$

b) (**optional**) Suppose that over the course of the interval $|t| < T/2$ the normal $\mathbf{N}$ varies adiabatically, but ultimately returns to its original value. Show that the total angle through which $\mathbf{S}$ consequently rotates is simply related to the solid angle swept out by $\mathbf{N}$.

At least two issues are worth pointing out here. First, we have an example of "anholonomy: the geometrical phenomenon in which nonintegrability causes variables [$\mathbf{S}$] to fail to return to their original values, when others, which drive them [$\mathbf{N}$], are altered round a cycle." Second, the phenomenon is *geometric*, in the sense that the (dynamical) system has a property (the change in the oscillation direction) that depends only on the shape of the path taken by $\mathbf{N}$ but not the details of when those values of $\mathbf{N}$ were visited. For a discussion of this and many related issues, see A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, 1989), especially one of the articles by M. V. Berry, p. 8 *et seq*.

**3) Photon polarization and the Poincaré sphere**: Consider a monochromatic beam of light of frequency $\omega$ propagating along the positive $z$ direction. As a function of position $\mathbf{r}$ and time $t$ the electric field $\mathbf{E}$ can be expressed as

$$\mathbf{E}(\mathbf{r},t) = \mathcal{E}\,\mathrm{Re}\,\mathbf{d}\,\exp i\omega\left((z/c)-t\right),$$

where $\mathcal{E}$ is a real, positive amplitiude and $\mathbf{d}$ is a vector in the $x$-$y$ plane having complex components $d_x$ and $d_y$ and normalized such that $\mathbf{d}^*\cdot\mathbf{d}=1$. The complex vector $\mathbf{d}$ describes the *polarization* of the beam.

a) Determine the nature of the polarization when
   i) $d_x$ and $d_y$ have the same phase but arbitrary magnitudes, and
   ii) $d_x = \pm i d_y$, *i.e.*, the components have the same magnitude but are out of phase.

b) In a word, state the nature of the polarization when neither of these conditions hold.

Our aim is to relate the complex polarization vector $\mathbf{d}$ to a two-component spinor $|\psi_{\mathbf{d}}\rangle$ and, thence, to a real, three-dimensional unit vector $\mathbf{r}_{\mathbf{d}}$. Here, $|\psi_{\mathbf{d}}\rangle$ is a particular linear combination of $|\pm\rangle$, a pair of orthnormal spinors spanning the space of spinors. In terms of $|\pm\rangle$ and the components of $\mathbf{d}$ we construct the spinor

$$|\psi_{\mathbf{d}}\rangle \equiv \frac{d_x + i d_y}{\sqrt{2}}|+\rangle + \frac{d_x - i d_y}{\sqrt{2}}|-\rangle\,.$$

Note that this is the eigenvalue $+1/2$ eigenspinor of the spin-half Hamiltonian

$$H = |\psi_{\mathbf{d}}\rangle\langle\psi_{\mathbf{d}}| - \frac{\mathrm{I}}{2},$$

where I is the identity operator. For a suitable real, three-dimensional (ordinary) unit vector $\mathbf{r}_{\mathbf{d}}$, this Hamiltonian can be expressed as a linear combination of Pauli operators:

$$H = \frac{1}{2}\,\mathbf{r}_{\mathbf{d}}\cdot\boldsymbol{\sigma}.$$

(Recall that any two-component spinor is a $+1/2$ eigenspinor of a Hamiltonian of this form.) In this way, we are connecting a polarization state $\mathbf{d}$ to a point $\mathbf{r}_{\mathbf{d}}$ on the unit sphere in three dimensions, *via* the spinor $|\psi_{\mathbf{d}}\rangle$ (or any rephasing of it). This sphere is called the Poincaré sphere; it is a useful way of representing states of polarization (and their rephasings).

c) Determine the nature of the polarization associated with
   i) the poles of the Poincaré sphere; and
   ii) the equator of the Poincaré sphere.

d) State the the nature of the polarizations associated the points on the Poincaré sphere that are neither at the poles nor at the equator.

This problem should shed some light on "the paradox of polarization being governed by the same algebra as quantum states of spin-1/2 particles... even though photons are quantum particles with spin 1..." [M. V. Berry, *J. Mod. Optics*, **34**, 1401 (1987); reprinted in A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, 1989)].

**4) Cartesian vectors – optional**: The aim of this question is to familiarise you with some of the notational conventions that we shall be using throughout the course, and also to give you some practice with the so-called summation convention, due to Einstein.

Consider a *cartesian basis*, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, for 3-dimensional vectors $\mathbf{x}$. Suppose that the basis vectors are normalised to unity and are mutually orthogonal (*i.e.*, they are orthonormal); then they possess the scalar products $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$, where $\mu$ and $\nu$ take the values 1, 2, or 3 (or $x$, $y$ or $z$). Here, $\delta_{\mu\nu}$ is the Kronecker symbol, which equals 1 when $\mu = \nu$, and equals 0 otherwise. You may think of this as the $(3 \times 3)$ identity matrix

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An arbitrary vector $\mathbf{x}$ is a linear combination of basis vectors, $\mathbf{x} = \sum_{\mu=1}^{3} x_\mu \mathbf{e}_\mu$, with the set of coefficients (called components) $\{x_\mu\}_{\mu=1}^{3}$. Notice that we can extract a component by taking the scalar product of a vector with the appropriate basis vector,

$$\mathbf{e}_\mu \cdot \mathbf{x} = \mathbf{e}_\mu \cdot \sum_{\nu=1}^{3} x_\nu \mathbf{e}_\nu = \sum_{\nu=1}^{3} \mathbf{e}_\mu \cdot \mathbf{e}_\nu x_\nu = \sum_{\nu=1}^{3} \delta_{\mu\nu} x_\nu = x_\mu.$$

It is very useful to adopt a convention, called summation convention, in which summation is implied over any twice-repeated indices; *e.g.*,

$$\mathbf{x} = \sum_{\mu=1}^{3} x_\mu \mathbf{e}_\mu \equiv x_\mu \mathbf{e}_\mu.$$

In true tensorial equations a given index, say $\mu$, never need occur more than twice. Singly occurring indices are called effective indices, whilst repeated indices are called dummy indices, and may be replaced by another index: *e.g.*, $\mathbf{x} = x_\nu \mathbf{e}_\nu = x_\mu \mathbf{e}_\mu$. Dummy indices are rather like dummy variables in integrals.

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal then their components are equal, *i.e.*, $a_\mu = b_\mu$. This follows from taking the scalar product of both sides of the equation $\mathbf{a} = \mathbf{b}$ with the basis vector $\mathbf{e}_\mu$. Notice that unrepeated indices balance throughout all terms of an equation. For example, if $\mathbf{a} + \mathbf{b} = \mathbf{c}$ then $a_\mu + b_\mu = c_\mu$. Indices are only considered repeated if they occur in the *same term*. For example, the equation $a_\mu = b_\mu$ contains one effective index and no repeated indices. Using $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$ and $\mathbf{e}_\mu \cdot \mathbf{x} = x_\mu$, and also the definition of $\delta_{\mu\nu}$, verify the following statements:

a-1) $\mathbf{x} \cdot \mathbf{x} = x_\mu x_\mu$

a-2) $\mathbf{x} \cdot \mathbf{y} = x_\mu y_\mu$

a-3) $\delta_{\mu\nu} \, \delta_{\nu\rho} = \delta_{\mu\rho}$

a-4) $a_\mu = a_\nu \delta_{\nu\mu}$

a-5) $\delta_{\mu\mu} = 3$.

Now consider scalar and vector fields, *i.e.*, scalar-valued functions, $f(\mathbf{x})$, and vector-valued functions, $\mathbf{g}(\mathbf{x}) = \mathbf{e}_\mu\, g_\mu(\mathbf{x})$, of a position vector, $\mathbf{x}$. For cartesian coordinates, the gradient operator $\nabla$ is defined by

$$\nabla \equiv \sum_{\mu=1}^{3} \mathbf{e}_\mu \, \frac{\partial}{\partial x_\mu} = \mathbf{e}_\mu \, \frac{\partial}{\partial x_\mu} = \mathbf{e}_\mu \, \partial_\mu$$

where, for convenience, we have written $\partial_\mu$ for $\partial/\partial x_\mu$.
Verify the following results:

b-1) $\nabla \cdot \mathbf{g}(\mathbf{x}) = \partial_\mu g_\mu(\mathbf{x})$
b-2) $\nabla f(\mathbf{x}) = \mathbf{e}_\mu \partial_\mu f(\mathbf{x})$
b-3) $(\mathbf{x} \cdot \nabla)\, f(\mathbf{x}) = x_\mu \, \partial_\mu f(\mathbf{x})$
b-4) $\nabla \cdot (\nabla f(\mathbf{x})) = \nabla^2 f(\mathbf{x})$ where $\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2 = \partial_\mu \partial_\mu$
b-5) $\nabla \cdot \mathbf{x} = \partial_\mu \, x_\mu = \delta_{\mu\mu} = 3$
b-6) $\nabla(\mathbf{x} \cdot \mathbf{x}) = 2\,\mathbf{x}$
b-7) $\nabla^2\,(\mathbf{x} \cdot \mathbf{x}) = 6$
b-8) $\nabla|\mathbf{x}| = \mathbf{x}/|\mathbf{x}|$
b-9) $\partial_\mu(x_\nu/|\mathbf{x}|) = (x^2 \delta_{\mu\nu} - x_\mu\, x_\nu)/x^3$ .
b-10) $\nabla^2\,(1/|\mathbf{x}|) = -4\pi\,\delta(\mathbf{x})$ [Hint: Apply the divergence theorem.]
b-11) $\nabla\,(\mathbf{x} \cdot \mathbf{g}(\mathbf{x})) = \mathbf{g}(\mathbf{x}) + \mathbf{e}_\mu x_\nu \partial_\mu g_\nu(\mathbf{x})$
b-12) $\nabla \cdot (\mathbf{x}\, f(\mathbf{x})) = 3\, f(\mathbf{x}) + (\mathbf{x} \cdot \nabla)\, f(\mathbf{x})$
b-13) For constant $\mathbf{h}$, $\oint_\Gamma d\mathbf{x} \cdot (\frac{1}{2}\mathbf{h}{\times}\mathbf{x}) = \pi \mathbf{h} \cdot \mathbf{n}$, where $\Gamma$ is a any circle of unit radius, and the unit vector $\mathbf{n}$ specifies the axis of the circle and the sense in which it is traversed. [Hint: Apply the Stokes theorem.]
b-14) $\nabla\, \exp(i\mathbf{k} \cdot \mathbf{x}) = i\mathbf{k}\, \exp(i\mathbf{k} \cdot \mathbf{x})$
b-15) $f(\mathbf{x} + \mathbf{a}) = f(\mathbf{x}) + (\mathbf{a} \cdot \nabla) f(\mathbf{x}) + \cdots = \mathrm{e}^{\mathbf{a} \cdot \nabla} f(\mathbf{x})$. [Note: This is a compact form of the multidimensional Taylor theorem.]

A large selection of further practice problems can be found in: M. R. Spiegel, *Vector Analysis* (McGraw-Hill, 1974), especially Chapters 4–6.

**5) More on cartesian vectors – optional**: The purpose of this question is two-fold. Firstly, we will investigate some of the properties of the vector product, denoted $\times$, and the related differential operator, curl, denoted $\nabla\times$. Secondly, we will solve the problems using summation convention so that we get some more practice with it. As with *Homework 0*, we consider an orthonormal basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, for 3-dimensional cartesian vectors, $\mathbf{x}$. The basis is said to be right-handed because

$$
\begin{aligned}
\mathbf{e}_1\times\mathbf{e}_2 &= \mathbf{e}_3 \\
\mathbf{e}_2\times\mathbf{e}_3 &= \mathbf{e}_1 \\
\mathbf{e}_3\times\mathbf{e}_1 &= \mathbf{e}_2.
\end{aligned}
$$

We can express these relationships much more compactly using the symbol $\epsilon_{\mu\nu\rho}$, known as the Levi-Civita symbol, or the antisymmetric third-rank tensor. This tensor takes on the following values in all cartesian coordinate systems:

$$
\epsilon_{\mu\nu\rho} = \begin{cases} +1, & \text{if } \mu\nu\rho = 123, 231, \text{ or } 312; \\ -1, & \text{if } \mu\nu\rho = 132, 213, \text{ or } 321; \\ 0, & \text{otherwise.} \end{cases}
$$

Notice that $\epsilon_{\mu\nu\rho}$ is totally antisymmetric, *i.e.*, its value changes sign whenever any pair of indices are exchanged, *e.g.*, $\epsilon_{123} = -\epsilon_{213} = 1$. This requirement forces $\epsilon_{\mu\nu\rho}$ to vanish whenever two or more of its indices are the same, *e.g.*, $\epsilon_{113} = 0$. This property is extremely useful, as we shall see, when it comes to proving certain results involving vector products and the curl operator.

In terms of $\epsilon_{\mu\nu\rho}$, the vector products between basis vectors become

$$
\mathbf{e}_\mu\times\mathbf{e}_\nu = \epsilon_{\mu\nu\rho}\,\mathbf{e}_\rho,
$$

where the implied summation on $\rho$ recovers the previously-given results for the cases $\mu \neq \nu$, and also includes results when $\mu = \nu$. Starting with these definitions, and the results from last week's homework if necessary, verify the following statements using summation convention:

a-1) $\epsilon_{\mu\nu\rho}\,\epsilon_{\mu\sigma\tau} = \delta_{\nu\sigma}\,\delta_{\rho\tau} - \delta_{\nu\tau}\,\delta_{\rho\sigma}$

a-2) $\epsilon_{\mu\nu\rho}\,\epsilon_{\mu\nu\tau} = 2\,\delta_{\rho\tau}$

a-3) $\epsilon_{\mu\nu\rho}\,\epsilon_{\mu\nu\rho} = 6$

a-4) $\mathbf{A}\times\mathbf{B} = A_\mu\,B_\nu\,\epsilon_{\mu\nu\rho}\,\mathbf{e}_\rho$

a-5) $(\mathbf{A}\times\mathbf{B})_\rho = A_\mu\,B_\nu\,\epsilon_{\mu\nu\rho}$

a-6) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \epsilon_{\mu\nu\rho}\,A_\mu\,B_\nu\,C_\rho$

a-7) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$

a-8) $\mathbf{A}\times\mathbf{A} = \mathbf{0}$

a-9) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

Now consider scalar and vector fields, *i.e.*, scalar-valued functions, $f(\mathbf{x})$, and vector-valued functions, $\mathbf{g}(\mathbf{x}) = \mathbf{e}_\mu\, g_\mu(\mathbf{x})$, of a position vector, $\mathbf{x}$. The curl operator, $\nabla\times$, operates on a vector field, $\mathbf{g}(\mathbf{x})$ to produce new vector field, denoted $\nabla\times\mathbf{g}(\mathbf{x})$. It is defined in the following way:

$$\nabla\times\mathbf{g}(\mathbf{x}) \equiv \sum_{\mu,\nu,\rho=1}^{3} \mathbf{e}_\mu\,\epsilon_{\mu\nu\rho}\,\frac{\partial}{\partial x_\nu} g_\rho(\mathbf{x}) = \mathbf{e}_\mu\,\epsilon_{\mu\nu\rho}\,\frac{\partial}{\partial x_\nu}\,g_\rho(\mathbf{x}) = \mathbf{e}_\mu\,\epsilon_{\mu\nu\rho}\,\partial_\nu\,g_\rho(\mathbf{x}).$$

Using these definitions, verify the following statements:

b-1) $\nabla\times\mathbf{x} = \mathbf{0}$

b-2) $\nabla\times(\mathbf{H}\times\mathbf{x}) = 2\,\mathbf{H}$, for constant $\mathbf{H}$

b-3) $\nabla\cdot(\nabla\times\mathbf{g}(\mathbf{x})) = 0$

b-4) $\nabla\times(\nabla f(\mathbf{x})) = \mathbf{0}$

b-5) $\nabla\times(f(\mathbf{x})\,\mathbf{g}(\mathbf{x})) = f\,\nabla\times\mathbf{g} + (\nabla f)\times\mathbf{g}$

b-6) $\nabla\times(\mathbf{g}(\mathbf{x})\times\mathbf{h}(\mathbf{x})) = \mathbf{g}\,\nabla\cdot\mathbf{h} - \mathbf{h}\,\nabla\cdot\mathbf{g} + (\mathbf{h}\cdot\nabla)\,\mathbf{g} - (\mathbf{g}\cdot\nabla)\,\mathbf{h}$