1) Quantifying entanglement: Consider a quantum system involving *two parties*, traditionally called Alice and Bob. We call such systems *bipartite*. They may each be, *e.g.*, a quantum spin (say, one spin-1 and one spin-5/2), but for now let us leave them arbitrary and denote orthonormal sets of vectors spanning Alice's Hilbert space $\{|\alpha\rangle\}$ and Bob's Hilbert space $\{|\beta\rangle\}$. Then arbitrary states $|\psi_A\rangle$ of Alice can be written $|\psi_A\rangle = \sum_{\alpha=1}^{n_a} A_\alpha |\alpha\rangle$; arbitrary states $|\psi_B\rangle$ of Bob can be written $|\psi_B\rangle = \sum_{\beta=1}^{n_b} B_\beta |\beta\rangle$.

The generic states of the composite bipartite system $|\Psi\rangle$ can be expressed in terms of the amplitude $\Psi_{\alpha\beta}$ as

$$|\Psi\rangle = \sum_{\alpha\beta} \Psi_{\alpha\beta}\, |\alpha\rangle \otimes |\beta\rangle.$$

The *unentangled* states of the composite bipartite system are the subset for which the amplitude $\Psi_{\alpha\beta}$ factorises: $\Psi_{\alpha\beta} = A_\alpha\, B_\beta$. The *entangled* states of the composite bipartite system are the subset for which the amplitude $\Psi_{\alpha\beta}$ does not factorise.

a) Show that if the amplitude $\Psi_{\alpha\beta}$ factorises then the state of the composite bipartite system $|\Psi\rangle$ factorises.

Let $\mathcal{O}_A$ and $\mathcal{O}_B$ be operators, respectively acting solely on Alice's and Bob's Hilbert spaces. (Think, *e.g.*, of the spin and position operators for a particle with spin.)

b) Show that for unentangled states the expectation value $\langle\Psi|\mathcal{O}_A\, \mathcal{O}_B|\Psi\rangle$ of the product operator $\mathcal{O}_A\, \mathcal{O}_B$ factorises into a product of expectation values, one involving each party. Briefly explain why we say that such states possess no quantum correlations between the parties.

Suppose we are concerned only with properties of one of the parties, say Alice. Rather than retain the full information $|\Psi\rangle$ (or, equivalently, the full density matrix $|\Psi\rangle\langle\Psi|$) describing the composite system, we may *trace out* Bob's Hilbert space. In this way, we develop a *reduced density matrix*

$$\rho_A \equiv \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|,$$

acting solely on Alice's Hilbert space. As long as we are only concerned with computing expectation values of operators on Alice, *i.e.*, operators of the form $\mathcal{O}_A \otimes \mathrm{I}_B$, where I_B is the identity on Bob's Hilbert space, this tracing presents no loss of options.

c) Show that if $|\Psi\rangle$ is unentangled then ρ_A represents a pure state.

Note that if $|\Psi\rangle$ is entangled then ρ_A is mixed. Thus, one strategy for quantifying the entanglement, at least for bipartite systems, is via the quantity $S_B \equiv -\mathrm{Tr}_B\, \rho_A\, \ln \rho_A$, familiar from the distinct setting of thermodynamics, which is non-negative, and zero only if ρ_A

describes a pure state. In summary, only if parties are unentangled does each of the parties individually inhabit a particular quantum state. Think about this in the setting of Bose-Einstein condensation: If a large number of bosonic atoms are in the same quantum state, are their fermionic constituent electrons (or neutrons, protons, quarks,...) all in the same quantum state? If so, how would this be consistent with the Pauli exclcusion principle?

As a strategy for *quantifying* the entanglement of a quantum state of a multi- (*i.e.* not necessarily bi-) partite system, we may invoke the notion of the accuracy with which a generic state can best be approximated by an unentangled one. If the best approximant is far from the state in question then that state must have been strongly entangled; if near then the state must have been weakly entangled.

Translating this idea into mathematics, we define the distance $\mathcal{D}$ between between a pair of normalised quantum states $|\Psi\rangle$ and $|\Phi\rangle$ via

$$\mathcal{D}(\Psi, \Phi) \equiv |\, |\Psi\rangle - |\Phi\rangle \,|,$$

where $|\cdots|$ indicates the norm of the enclosed vector in Hilbert space. Then, for a particular quantum state $|\Psi\rangle$ we can define its *geometric measure of entanglement* $\mathcal{D}(\Psi)$ to be the distance from $|\Psi\rangle$ to the closest unentangled normalised approximant:

$$\mathcal{D}(\Psi) = \min_{\Phi \in \Sigma_U} \mathcal{D}(\Psi, \Phi),$$

where Σ_U is the set of normalised unentangled states. The distance $\mathcal{D}(\Psi)$ measures the entanglement of $|\Psi\rangle$ in the sense that small/large $\mathcal{D}(\Psi)$ means weak/strong entanglement.

d) Show that we can equivalently define $\mathcal{D}(\Psi)$ by minimising not $|\, |\Psi\rangle - |\Phi\rangle \,|$ but rather $\frac{1}{2}|\, |\Psi\rangle - |\Phi\rangle \,|^2$ and, therefore, by maximising the cosine of the *angle* ϑ between a pair of normalised quantum states $|\Psi\rangle$ and $|\Phi\rangle$, this being defined via

$$\cos\vartheta(\Phi, \Psi) \equiv \mathrm{Re}\,\langle\Phi|\Psi\rangle.$$

Hence, once can equivalently define the geometric measure of entanglement of the quantum state $|\Psi\rangle$ to be $2|\sin(\vartheta(\Psi)/2)|$, where

$$\cos\vartheta(\Psi) = \max_{\Phi \in \Sigma_U} \cos\vartheta(\Phi, \Psi).$$

This angle $\vartheta(\Psi)$, measures the entanglement of $|\Psi\rangle$, in the sense that small/large $\vartheta(\Psi)$ [or, equivalently, $2|\sin(\vartheta(\Psi)/2)|$] means weak/strong entanglement.

e) Show that $\vartheta(\Psi)$ can also be interpreted via the energy expectation value in the best variational ground state of the auxilliary Hamiltonian $\mathcal{H} \equiv -|\Psi\rangle\langle\Psi|$ over the set Σ_U.

Consider the special case of *bipartite* quantum systems, and let the state whose entanglement is being quantified be characterised by the amplitides $\Psi_{\alpha\beta}$ mentioned in the introduction.

f) By varying $\cos\vartheta(\Phi,\Psi)$ with respect to the normalised unentangled state Φ, show that the amplitudes A_α and B_β characterising the best unentangled approximant obey the *linear* $(n_a+n_b)\times(n_a+n_b)$ eigenproblem

$$\begin{pmatrix} \mathbf{0} & \mathbf{X} \\ \mathbf{X}^\dagger & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{B}^* \\ \mathbf{A} \end{pmatrix} = \Lambda \begin{pmatrix} \mathbf{B}^* \\ \mathbf{A} \end{pmatrix},$$

where $\mathbf{X}$ is the $n_a \times n_b$ matrix with elements $\mathbf{X}_{\alpha\beta} = \Psi_{\alpha\beta}$, $\mathbf{A}$ is the n_a-entried column matrix with elements A_α, and $\mathbf{B}$ is the n_b-entried column matrix with elements B_β.

g) Show that the largest of the *entanglement eigenvalues* Λ warrants its name by having the value $\cos\vartheta(\Psi)$.

h) Show that by "squaring" the eigenproblem one arrives at the block-diagonal form

$$\begin{pmatrix} \mathbf{X}\cdot\mathbf{X}^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^\dagger\cdot\mathbf{X} \end{pmatrix} \begin{pmatrix} \mathbf{B}^* \\ \mathbf{A} \end{pmatrix} = \Lambda^2 \begin{pmatrix} \mathbf{B}^* \\ \mathbf{A} \end{pmatrix}.$$

Explain how the eigenproblem thus reduces to one no harder than the smaller of $n_a \times n_a$ and $n_b \times n_b$. Note that the matrices $\mathbf{X}\cdot\mathbf{X}^\dagger$ and $\mathbf{X}^\dagger\cdot\mathbf{X}$ are square and Hermitian, and can be shown to be isospectral (up to zero modes).

Now consider the general case of P-partite entanglement, in which case the generic state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = \sum_{\alpha_1=1}^{n_1}\sum_{\alpha_2=1}^{n_2}\cdots\sum_{\alpha_P=1}^{n_P} \Psi_{\alpha_1\alpha_2\cdots\alpha_P}\, |E^{(1)}_{\alpha_1}\rangle \otimes |E^{(2)}_{\alpha_2}\rangle \otimes \cdots \otimes |E^{(P)}_{\alpha_P}\rangle,$$

where $\{|E^{(p)}_{\alpha_p}\rangle\}_{p=1}^{n_p}$ is the Hilbert-space basis for the states of party p (with $p = 1,2,\ldots,P$). The general unentangled state has the form

$$|\Phi\rangle = \sum_{\alpha_1=1}^{n_1}\sum_{\alpha_2=1}^{n_2}\cdots\sum_{\alpha_P=1}^{n_P} \varphi^{(1)}_{\alpha_1}\varphi^{(2)}_{\alpha_2}\cdots\varphi^{(P)}_{\alpha_P}\, |E^{(1)}_{\alpha_1}\rangle \otimes |E^{(2)}_{\alpha_2}\rangle \otimes \cdots \otimes |E^{(P)}_{\alpha_P}\rangle,$$

in which the individual components $\{\varphi^{(p)}_{\alpha_p}\}_{p=1}^{n_p}$ of the factorised amplitude can themselves be taken to be normalised, *i.e.*, $\sum_{\alpha_p=1}^{d_p} |\varphi^{(p)}_{\alpha_p}|^2 = 1$ (for $p = 1,2,\ldots,P$).

i) For the multipartite (*i.e.* $P \geq 3$) case, derive the the *nonlinear eigenproblem* determining both the best unentangled approximant and the corresponding entanglement eigenvalue.

2) Supersymmetry (after G. Junker, Supersymmetric Methods in Quantum and Statistical Physics) – optional: Consider quantum systems involving a Hamiltonian H, along with N $(= 1, 2, \ldots, N)$ hermitian operators Q_i $(i = 1, 2, \ldots, N)$, acting on a Hilbert space $\mathcal{H}$. If it is true that the anticommutation relations

$$\{Q_i, Q_j\} = H\,\delta_{ij}$$

hold for all i and j then we call the system *supersymmetric* (SUSY); we call the Q_i *supercharges* and the Hamiltonian H the *SUSY Hamiltonian*. The symmetry characterised by these anticommutation relations (or *superalgebra*) is called N-extended supersymmetry.

a) Show that the superalgebra guarantees that

$$H = 2\,Q_1^2 = 2\,Q_2^2 = \cdots = 2\,Q_N^2 = \frac{2}{N}\sum_{i=1}^{N} Q_i^2,$$

i.e., the supercharges are "square roots" of the Hamiltonian.

b) Show that $[H, Q_i] = 0$ (for $i = 1, 2, \ldots, N$). That is, if the supercharges have no explicit time dependence then they are conserved.

c) Show that the hamiltonian does not have any negative eigenvalues.

It is useful to introduce the notions of *good* and *broken* SUSY. If the energies of the ground states the system are zero we say that SUSY is good; if it is larger than zero we say that SUSY is broken.

d) Show that if SUSY is good then all ground states are annihiliated by all supercharges.

e) Show that if SUSY is broken then at least one supercharge fails to annihilate at least one ground state.

Consider the Pauli Hamiltonian, which describes the dynamics of a spin-half particle of mass m and charge e moving in, say, three dimensions in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$:

$$H_{\mathrm{P}} \equiv \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}|^2 - \frac{e\hbar}{2mc}\,\mathbf{B}\cdot\sigma.$$

Here, $\mathbf{A}$ is the vector potential and the gyromagnetic ratio is two.

f) Show that the system is $N = 1$ supersymmetric, with supercharge

$$Q_1 \equiv \frac{1}{\sqrt{4m}}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)\cdot\sigma.$$

It is a curious fact that supersymmetry constrains the gyromagnetic ratio to be ± 2, absent any considerations of special relativity. (Recall that a gyromagnetic ratio of two is usually taken to follow from the Dirac equation.)

Witten's $N = 2$ SUSY quantum mechanics describes the dynamics of a spin-half particle of mass m moving in one dimension in a potential. The two supercharges are

$$\begin{aligned} Q_1 &= \frac{1}{\sqrt{2}}\left(\frac{p}{\sqrt{2m}}\,\sigma_1 + \Phi(x)\,\sigma_2\right), \\ Q_2 &= \frac{1}{\sqrt{2}}\left(\frac{p}{\sqrt{2m}}\,\sigma_2 - \Phi(x)\,\sigma_1\right), \end{aligned}$$

where Φ, which is known as the SUSY potential (not the superpotential), is assumed to be continuous.

g) Show that the Hamiltonian of the Witten model is

$$H_{\mathrm{W}} \equiv \left(\frac{p^2}{2m} + \Phi(x)^2\right) + \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)\,\sigma_3,$$

where the prime indicates a derivative. Note that in the eigenbasis of σ_3 the Hamiltonian is block diagonal:

$$\begin{aligned} H_{\mathrm{W}} &= \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \\ H_\pm &\equiv \frac{p^2}{2m} + \Phi(x)^2 \pm \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)\,. \end{aligned}$$

3) Variational approach to the particle in a box – optional: *Shankar*, 16.1.2.

4) Variational approach to the attractive potential (after Shankar, 16.1.3): Consider a particle of mass m moving in one spatial dimension. For the attractive delta-function potential $V(x) = -aV_0\,\delta(x)$ use a Gaussian trial wave function to calculate an upper bound on the ground state energy. Compare your answer to the exact answer, $-ma^2V_0^2/2\hbar^2$.