1) Coherent states: In quantum optics, the classical limit of quantum mechanics, quantum field theory and quantum statistical mechanics, there is a very important set of states, known as *coherent states*, which may be constructed from the harmonic oscillator energy eigenstates $\{|n\rangle\}$. Each coherent state is labelled by a single complex number z; it is denoted $|z\rangle$, and defined by $|z\rangle \equiv \exp(za^\dagger)|0\rangle$, where the ket $|0\rangle$ is the harmonic oscillator ground state.

a) Show that the coherent state $|z\rangle$ is an eigenstate of the annihilation operator a. What is its eigenvalue?
b) Evaluate $\langle z|a^\dagger$ in terms of $\langle z|$ and z^*.
c) Show that $(d/dz)|z\rangle = a^\dagger|z\rangle$.
d) Show that the inner product of two coherent states is given by $\langle z|z'\rangle = \exp(z'z^*)$.

Consider the function of two complex variables $f(z, z')$. The operator obtained by inserting $a^\dagger$ for z and a for z' is generally ambiguous because the order of the operators is not specified. For any f we can unambiguously define an operator $f(a^\dagger, a)$, in which the operator ordering ambiguity is resolved by the prescription that, in any product, all annihilation operators lie to the right of all creation operators. Operators of this form are called *normal ordered* operators.

e) Show that for normal ordered operators we have $\langle z|f(a^\dagger, a)|z'\rangle = e^{z^*z'} f(z^*, z')$.
f) Show that the identity operator I may be written as

$$I = \frac{1}{\pi}\int d\mathrm{Re}z\, d\mathrm{Im}z\, |z\rangle\, e^{-|z|^2}\langle z|,$$

where Rez and Imz denote the real and imaginary parts of the complex number z, and the integral extends over the entire complex plane.

g) Show that the coherent states form an overcomplete set, *i.e.*, that there are non-trivial linear combinations of coherent states that vanish.
h) Show that the trace of an operator, $\mathrm{Tr}\, C$, can be written in terms of coherent states as

$$\mathrm{Tr}\, C = \frac{1}{\pi}\int d\mathrm{Re}z\, d\mathrm{Im}z\, e^{-|z|^2}\,\langle z|C|z\rangle.$$

i) At time $t = 0$ the system is prepared in the coherent state $|z\rangle$. Suppose that the system evolves according to the hamiltonian $H = \hbar\omega a^\dagger a$. Determine the state of the system at the later time t? Do coherent states remain coherent states under time-evolution?
j) [Parts (j) to (m) are **optional**] Compute the expectation values of q and p in the state into which $|z\rangle$ evolves at time t.

k) Compare the expectation values obtained in part (j) with those obtained in energy eigenstates of the oscillator. Compare the time-dependence of the coherent-state expectation values of q and p with those of the classical oscillator.

l) Compute the uncertainties in q and p at time t. Comment on their product.

m) Compare coherent states with the oscillator ground state.

2) Entanglement for two spin-half freedoms: Entanglement is the name given by Schrödinger in 1935 to the quintessentially quantum-mechanical phenomenon in which "the best possible knowledge of the whole [of a system] does not include the best possible knowledge of its parts." The name *entanglement* is a loose translation from the German *Verschränkung*, which means interlaced, as in the folding of one's arms.

Consider a pair of distinguishable spin-half freedoms, 1 and 2, in the pure state

$$|\Psi\rangle = \cos\alpha |+-\rangle + \sin\alpha |-+\rangle.$$

Here, α is a (real) parameter in the range $0 \le \alpha < 2\pi$, and the state vectors $|\sigma_1 \sigma_2\rangle$ $[\equiv |\sigma_1\rangle \otimes |\sigma_2\rangle_2]$ are product vectors in which the z projections S_{z1} and S_{z2} of the two spin-operators $\mathbf{S}_1$ $(\equiv \frac{1}{2}\hbar\boldsymbol{\sigma}_1)$ and $\mathbf{S}_2$ $(\equiv \frac{1}{2}\hbar\boldsymbol{\sigma}_2)$ have the sharp values σ_1 $(\equiv \pm)$ and σ_2 $(\equiv \pm)$.

a) For what values of α is $|\Psi\rangle$ unentangled?

b) Compute the expectation value of $\boldsymbol{\sigma}_1$ in the state $|\Psi\rangle$. For what values of α could this result have been obtained if the state of the first spin were described by a state vector in the Hilbert space of a single spin-half freedom?

c) Given that the two-spin system is in the state $|\Psi\rangle$, by tracing over the states of the second spin, construct the density matrix

$$\rho_1 \equiv \mathrm{Tr}_2 \, |\Psi\rangle \, \langle\Psi|$$

describing the first spin. Determine the values of α, if any, for which ρ_1 describes a pure state. For what values of α is the description of spin 1 the best possible description of a spin-half system admitted by quantum mechanics?

d) Let $\langle \cdots \rangle$ denote expectation values taken in the state $|\Psi\rangle$. Compute the following correlator between quantum fluctuations in the state $|\Psi\rangle$:

$$\langle (\sigma_{1z} - \langle\sigma_{1z}\rangle) \, (\sigma_{2z} - \langle\sigma_{2z}\rangle) \rangle \, .$$

For what values of α does this correlator vanish?

3) The Einstein-Podolsky-Rosen paradox: Consider a composite system comprising a pair of distinguishable spinlesss particles, 1 and 2, moving in one spatial dimension. Consider the following pure state describing them, expressed in terms of their position eigenstates:

$$|\Psi\rangle = \int dy\, |y + a\rangle_1 \otimes |y\rangle_2.$$

a) Calculate the two-particle real-space wave function $\Psi(x_1, x_2)$, ignoring normalisation.
b) Suppose that the position of particle 1 is observed when the composite system is in the state $|\Psi\rangle$. What can you say about the likelihood of finding particle 1 to be at any particular position?
c) Suppose instead that the composite system is in the state $|\Psi\rangle$ and that the position of particle 2 is found to be at x_2. What result can you expect the immediate subsequent measurement of the position of particle 1 to yield?
d) Express $|\Psi\rangle$ in terms of the momentum eigenstates $|p_1\rangle_1$ and $|p_2\rangle_2$. Calculate the two-particle momentum-space wave function $\widetilde{\Psi}(p_1, p_2)$, ignoring normalisation.
e) Suppose that the momentum of particle 1 is observed when the composite system is in the state $|\Psi\rangle$. What can you say about the likelihood of finding particle 1 to have any particular momentum?
f) Suppose instead that the composite system is in the state $|\Psi\rangle$ and that the momentum of particle 2 is found to be at p_2. What result can you expect the immediate subsequent measurement of the momentum of particle 1 to yield?

As you have just shown, in the entangled state $|\Psi\rangle$ measurement of the position of particle 2 creates certainty about the position of particle 1, and measurement of the momentum of particle 2 creates certainty about the momentum of particle 1. This deeply troubled Einstein, Podolsky and Rosen, who understood that quantum mechanics denies the possibility of our having certain knowledge about both the position and momentum of a particle. Reflect on this issue, and think about ideas about reality we give up when we side with quantum mechanics.