1) Block diagonal matrices, time evolution and conservation laws: Suppose a system is governed by some complicated hamiltonian that we are unable to diagonalise. Can we still make specific statements about its time-evolution? This question will explain how we can.

Suppose that you can identify (usually through observing a symmetry) a Hermitian operator A that commutes with the hamiltonian H, *i.e.*, $[A, H] = 0$. Suppose A to be sufficiently simple that you are able to find a complete set of kets $|a, \gamma\rangle$ that are eigenkets of A with eigenvalue a. The label γ distinguishes A-degenerate kets.

a) Show that H is block diagonal in the $|a, \gamma\rangle$-basis.

b) How are the block sizes related to the degeneracies of A?

c) Sketch a typical block-diagonal matrix $\langle a', \gamma'|H|a, \gamma\rangle$.

d) Show that the product of two matrices with a common block structure is also a matrix with that block structure.

e) Show that the matrix elements of a function f of a matrix R has the same block structure as the matrix R. For a time-independent hamiltonian H, does the time-evolution operator $U(t) \equiv \exp(-iHt/\hbar)$ have the same block structure as H?

f) Consider a state $|\psi_0\rangle$ that happens to be a linear combination of states drawn from the same A-multiplet, *i.e.*,

$$|\psi_0\rangle = \sum_{\gamma=1}^{n(a)} C_\gamma |a, \gamma\rangle.$$

What values of A may result from observations of A?

g) What would be the state immediately after an observation of A on the state $|\psi_0\rangle$?

h) Suppose, now, that $|\psi_0\rangle$ evolves in time according to the hamiltonian H of part (a), which commutes with A. By considering the block structure of H in the A-basis, and the structure of $|\psi_0\rangle$ in the same basis, show that time-evolution simply mixes the A-degenerate states.

i) $|\psi_0\rangle$ evolves into the state $|\psi_t\rangle$ under the action of H. What values of A may result from observations on $|\psi_t\rangle$ at later times?

j) Can you say that A is conserved under the time evolution generated by H?

k) Establish this result by relating the ket $AU(t)|\psi_0\rangle$ to the ket $U(t)A|\psi_0\rangle$.

l) Briefly discuss the conservation of A using Heisenberg's equation of motion.

m) Consider the three-state system $|i\rangle$, where $i = 1, 2, 3$. The matrix elements of the hamiltonian for this system in the $\{|i\rangle\}$-basis are given by

$$\langle i|H|j\rangle = \begin{pmatrix} 0 & -i & \delta \\ i & 0 & 0 \\ \delta & 0 & 1 \end{pmatrix},$$

with real δ. The observable A has matrix elements

$$\langle i|A|j\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Without explicitly evaluating the commutator, but instead by examining the block structure, do you expect these matrices to commute? Why?

n) In your answer to part (m), what changes when $\delta = 0$? Why?

2) Harmonic oscillator: In this question we will consider the one-dimensional harmonic oscillator described by the hamiltonian

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2.$$

It is very useful to introduce the annihilation operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + i\sqrt{\frac{1}{2m\hbar\omega}}P,$$

and its adjoint, the creation operator

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2m\hbar\omega}}P.$$

a) Compute the commutator $[a, a^\dagger]$ given the commutation relation $[X, P] = i\hbar I$.

b) Show that in terms of the creation and annihilation operators the hamiltonian H becomes $H = \hbar\omega(a^\dagger a + 1/2)$.

c) Evaluate the commutators $[H, a]$ and $[H, a^\dagger]$. Use your results to explain why creation and annihilation operators are suitable names for the operators $a^\dagger$ and a.

d) Write down the Heisenberg equations of motion for the Heisenberg operators $a_H(t)$ and $a_H^\dagger(t)$. Using the results for the relevant commutators from part (c), solve these equations of motion to obtain explicit expressions for the Heisenberg operators $a_H(t)$ and $a_H^\dagger(t)$ in terms of the Schrödinger operators a and $a^\dagger$.

A certain Green function $G(t_2, t_1)$ is defined as the expectation value in the harmonic oscillator ground state of the time-ordered product of Heisenberg creation and annihilation operators, in the following way:

$$G(t_2, t_1) \equiv i\langle 0|\mathrm{T}[a_H(t_2)a_H^\dagger(t_1)]|0\rangle.$$

The time-ordered product of two operators $\mathrm{T}[A(t_2)B(t_1)]$ is defined to be the product of the operators, but with the order of the terms in the product chosen so that the operator with the later time-argument is the left-hand factor of the product, *i.e.*,

$$\mathrm{T}[A(t_2)B(t_1)] \equiv \theta(t_2 - t_1)A(t_2)B(t_1) + \theta(t_1 - t_2)B(t_1)A(t_2).$$

e) Use your solution to part (d) to compute G.

f) By differentiating the definition of $G(t_2, t_1)$, given above, with respect to the time t_2, construct an equation of motion for $G(t_2, t_1)$. Show that your answer to part (e) solves the equation of motion.

g) Recall that the annihilation operator, a, annihilates the harmonic oscillator ground state, *i.e.*, $a|0\rangle = 0$. By resolving this equation on to the position eigenbasis, and recalling the form of the operators X and P in that basis, construct a *first order* ordinary differential equation for the ground state wave function $\langle x|0\rangle$. Solve this equation, and hence construct the normalised ground state wave function for the one-dimensional harmonic oscillator.

h) Suppose the quadratic potential energy is augmented by a quartic term gX^4. Compute the expectation value of this operator in the states $|n = 0\rangle$ and $|n = 1\rangle$.
[Hint: Although there are many ways to perform this computation, one of the most useful ways is the following. First, write the bra and ket in terms of the ground state and creation and annihilation operators. Second, write the operator gX^4 in terms of creation and annihilation operators. Third, notice that of all the terms that appear in the expansion of X^4, the only terms that survive (*i.e.*, do not vanish) are those having the same number of creation and annihilation operators. Finally, use the commutation relation between creation and annihilation operators to re-order the operators with the aim of moving the annihilation operators to the right and the creation operators to the left. This process is called *normal ordering*. Its virtue is that as it proceeds many terms will vanish because $a|n = 0\rangle = 0$ and $\langle n = 0|a^\dagger = 0$.]