1) Exponential operators: In this question we shall consider a function of the two operators Λ and Ω. First, we shall specialise to the case in which the operators are such that their commutator is proportional to the identity, $[\Lambda, \Omega] = d\,\mathrm{I}$. Often you will see this written $[\Lambda, \Omega] = d$; a situation described by the statement 'the commutator is a c-number (or commuting or classical number)'. By following the strategy below, we will prove that for operators whose commutator is a c-number: $\exp\Lambda\, \exp\Omega = \exp\left(\Lambda + \Omega + \frac{1}{2}[\Lambda, \Omega]\right)$.

a) Check the consistency of this equation by expanding the exponential functions retaining terms of quadratic and lower order in the operators Λ and Ω. Is the theorem consistent to this order?

b) Introduce the c-number μ and the operator-valued functions

$$\begin{aligned} f(\mu) &\equiv \mathrm{e}^{\mu\Lambda}\,\mathrm{e}^{\mu\Omega}, \\ g(\mu) &\equiv \mathrm{e}^{\mu(\Lambda+\Omega)+\frac{1}{2}\mu^2[\Lambda,\Omega]}. \end{aligned}$$

Evaluate $df/d\mu$ (remembering that ordering of operator matters).

c) Evaluate $dg/d\mu$.

d) By expanding $\exp(\mu\Omega)$, and hence evaluating $[\Lambda, \exp(\mu\Omega)]$, prove that f and g satisfy the same differential equation, *viz.*,

$$\frac{dh}{d\mu} = h(\mu)\ (\mu d + \Lambda + \Omega).$$

e) Is $f(0)$ equal to $g(0)$? Hence prove that $f(\mu) = g(\mu)$. By setting $\mu = 1$ we have the desired result.

Hint: To evaluate $[\Lambda, \exp(\mu\Omega)]$, introduce $J_n\,\Omega^{n-1} \equiv [\Lambda, \Omega^n]$.
Then use $[M, NP] = N[M, P] + [M, N]P$ to find an equation for J_n.

For the remainder of this question we shall consider the general case, when $[\Lambda, \Omega]$ is *not* a c-number. Now the general result is not valid, but it is approximately true for small μ. To see this, consider $f(\mu)$ and $g(\mu)$ as defined in part (a).

f) By expanding f and g as power series in μ, find the order to which f and g remain equal.

2) Simultaneous Diagonalisation – optional: Shankar, 1.8.10, page 46.

3) Logarithm operators. The general strategy for evaluating the matrix elements of a function of a hermitean operator in an arbitrary orthonormal basis is:

- transform to a basis in which the operator is diagonal;
- replace the eigenvalues by the function of the eigenvalues;
- revert to the original basis.

Suppose that the arbitrary orthonormal basis is $\{|a_i\rangle\}$, the function is f, and the operator is Ω (with eigenvalues $\{\omega_i\}$ and eigenvectors $\{|\omega_i\rangle\}$). Then, by applying this strategy we find that

$$\begin{aligned}\langle a_i|f(\Omega)|a_j\rangle &= \sum_{kl}\langle a_i|\omega_k\rangle\,\langle\omega_k|f(\Omega)|\omega_l\rangle\,\langle\omega_l|a_j\rangle \\ &= \sum_{kl}\langle a_i|\omega_k\rangle\,\langle\omega_k|\omega_l\rangle\,f(\omega_l)\,\langle\omega_l|a_j\rangle \\ &= \sum_{k}\langle a_i|\omega_k\rangle\,f(\omega_k)\,\langle\omega_k|a_j\rangle.\end{aligned}$$

Apply this strategy to the following example from $\mathcal{V}^{(2)}(\mathcal{R})$, with $f(x) = \ln(1-x)$.

a) The matrix representing Ω in the $\{|a_i\rangle\}$ basis is given by

$$\begin{pmatrix}\langle a_1|\Omega|a_1\rangle & \langle a_1|\Omega|a_2\rangle \\ \langle a_2|\Omega|a_1\rangle & \langle a_2|\Omega|a_2\rangle\end{pmatrix} = \begin{pmatrix} q\cos 2\theta & q\sin 2\theta \\ q\sin 2\theta & -q\cos 2\theta\end{pmatrix}.$$

Evaluate the matrix $\langle a_i|f(\Omega)|a_j\rangle$.

The matrix representing the operator Ψ in the $\{|a_i\rangle\}$ basis is given by

$$\begin{pmatrix}\langle a_1|\Psi|a_1\rangle & \langle a_1|\Psi|a_2\rangle \\ \langle a_2|\Psi|a_1\rangle & \langle a_2|\Psi|a_2\rangle\end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & -q\end{pmatrix}.$$

b) By using the two methods outlined below, show, *in two different ways*, that

$$\begin{pmatrix}\langle a_1|\ln(\mathrm{I}-\Psi)|a_1\rangle & \langle a_1|\ln(\mathrm{I}-\Psi)|a_2\rangle \\ \langle a_2|\ln(\mathrm{I}-\Psi)|a_1\rangle & \langle a_2|\ln(\mathrm{I}-\Psi)|a_2\rangle\end{pmatrix} = \begin{pmatrix}\ln(1-q) & 0 \\ 0 & \ln(1+q)\end{pmatrix}.$$

Method 1: Recognise that, in this basis, Ψ is already diagonal. Thus, by series expansion of $\ln(1-z)$, we can replace the eigenvalues of Ψ by the functions of the eigenvalues.
Method 2: Recognise that in the series expansion of $\ln(\mathrm{I}-\Psi)$ only two matrices occur,

$$q\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \qquad \text{and} \qquad q\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.$$

By regrouping the terms in the expansion, prove the desired result.

4) The resolvent operator: In this question we shall consider the connection between the density of eigenvalues $\{\omega_i\}$ of a hermitean operator Ω and a different operator called the resolvent operator (or Green operator) $R(z)$.

The density of eigenvalues is a function $n(\omega)$ that is a histogram of the eigenvalues of the operator Ω:

$$n(\omega) \equiv \sum_i \delta(\omega - \omega_i).$$

Often, such a function is called the *density of states*.

a) Show that the mean eigenvalue $\sum_i \omega_i / \sum_i 1$ is given by

$$\int d\omega\, \omega\, n(\omega) \Big/ \int d\omega\, n(\omega).$$

We introduce the resolvent operator $R(z)$, which depends on the complex variable z and on the operator Ω:

$$R(z) \equiv (z - \Omega)^{-1} \equiv \frac{1}{z - \Omega}.$$

Think of the resolvent operator as a function of the operator Ω and the complex variable z. Notice the convention that the operator zI is written z.

b) By going to the $\{|\omega_i\rangle\}$ representation, in which Ω is diagonal [and by using part (m) of question (3) of Homework 3] show that

$$n(\omega) = -\lim_{\epsilon \to 0} \pi^{-1}\, \mathrm{Im}\, \mathrm{Tr}\, R(\omega + i\epsilon),$$

where the *trace* operation, Tr, is defined via $\mathrm{Tr}\, A \equiv \sum_j \langle \omega_j | A | \omega_j \rangle$. Thus, we see that the resolvent operator encodes the density of states. Usually the operator Ω is the hamiltonian, in which case $n(\omega)$ is the density of energy levels. From the density of energy levels one can compute, for example, the specific heat capacity of the system.

c) Suppose that Ω consists of a dominant piece, Ω_0, and a small perturbation, $t\Omega_1$, *i.e.*,

$$\Omega = \Omega_0 + t\Omega_1.$$

Then the resolvent operator of the full system is $R(z) = (z - \Omega)^{-1}$, whilst the resolvent operator of the unperturbed system is $R_0(z) = (z - \Omega_0)^{-1}$. Show that $R(z)$ satisfies Dyson's equation,

$$R(z) = R_0(z) + tR_0(z)\Omega_1 R(z).$$

d) By iteratively solving Dyson's equation, write down the first three terms in an expansion of $R(z)$ in powers of t, in terms of $R_0(z)$ and Ω_1.

It is often the case in quantum mechanics that the small perturbation prevents exact diagonalisation. In such cases, the Dyson equation provides a compact scheme for computing perturbative corrections to $R(z)$, in powers of the strength of the perturbation t.

5) Theta function – optional: Shankar, 1.10.3, page 63.