1) Operators and matrix elements:

a) Show that if $\Lambda^{-1}\Lambda = I$ and $\Omega^{-1}\Omega = I$ then $(\Omega\Lambda)^{-1} = \Lambda^{-1}\Omega^{-1}$.

b) Suppose that $\{|i\rangle\}$ forms an orthonormal basis. Show that if $\Omega|i\rangle = \sum_j \Omega_{ji}|j\rangle$ then $\langle k|\Omega|l\rangle = \Omega_{kl}$.

c) Show that if, in addition, $\Lambda|i\rangle = \sum_j \Lambda_{ji}|j\rangle$ then $\langle k|\Lambda\Omega|l\rangle = \sum_m \Lambda_{km}\Omega_{ml}$.

Consider two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$.

d) Write down the adjoint of the operator $\Theta = |\psi\rangle\langle\psi|$?

e) Write down the adjoint of the operator $\Xi = |\psi\rangle\langle\psi| + i|\phi\rangle\langle\phi|$?

f) In the basis $\{|i\rangle\}$ the operator Σ is represented by the matrix

$$\begin{pmatrix} 0 & i \\ 1+i & 2i \end{pmatrix}.$$

Construct the matrix that represents the adjoint operator $\Sigma^\dagger$ in the same basis?

g) By using the relationship $(\Gamma^\dagger)_{ij} = (\Gamma_{ji})^*$ show that for two arbitrary kets, $|\psi\rangle$ and $|\phi\rangle$, we have $\langle\psi|\Theta^\dagger|\phi\rangle = \langle\phi|\Theta|\psi\rangle^*$.

2) Projection operators: Operators are called projection operators if they are idempotent, *i.e.*, $\Omega^2 = \Omega$. Once you have acted on a ket with an idempotent operator, repeated action with such an operator will no longer change the ket; the operator has lost its potency. Consider the operator $P^{(i)} \equiv |i\rangle\langle i|$, where no sum on i is implied, and $|i\rangle$ is one particular ket from an orthonormal basis.

a) Show that $P^{(i)}$ is an example of an idempotent operator, *i.e.*, ${P^{(i)}}^2 = P^{(i)}$.

b) Calculate the eigenvalues and associated eigenvectors of $P^{(i)}$?

c) Show that $P^{(i)}P^{(j)} = \delta_{ij}\, P^{(i)}$.

d) Show that the eigenvalues of *any* projection operator can only take the values 0 or 1.

e) Evaluate the matrix elements of $P^{(i)}$ in the $|i\rangle$-basis.

f) For any operator A, $\mathrm{Tr}A \equiv \sum_k A_{kk}$. Evaluate $\mathrm{Tr}P^{(i)}$.

g) How many non-zero eigenvalues does $P^{(i)}$ have?

h) Is $P^{(i)}$ hermitean?

3) Exercises from Shankar – optional:

a) Exercise 1.8.1 on page 41.

b) Exercise 1.8.2 on page 41.

4) A linear vector space – optional: Consider the LVS $\mathcal{V}^{(n)}(\mathcal{C})$.

a) State its dimensionality?

You are given a complete orthonormal set of vectors $\{|i\rangle\}$ for $\mathcal{V}^{(n)}(\mathcal{C})$.

b) Write a expression for an arbitrary vector $|\phi\rangle$, in terms of expansion coefficients ϕ_i and the basis kets $|i\rangle$.

c) As the basis is orthonormal, state the value of $\langle i|j\rangle$?

d) Give $\langle i|\phi\rangle$ in terms of your general expression for $|\phi\rangle$?

e) In terms of your general expression for $|\phi\rangle$, write down an expression for $\langle\phi|$?

f) In terms of your expansion coefficients, compute $\langle\phi|\phi\rangle$.

g) By considering its action on an arbitrary ket, show that the operator $\sum_i |i\rangle\langle i|$ is the identity operator I.

h) Compute the matrix elements between the states $|k\rangle$ and $|l\rangle$ of the operator $\sum_{ij} |i\rangle\Omega_{ij}\langle j|$.

i) By considering precisely the representation of operators found in part (g), show that the ik^{th} matrix element of the product operator $\Omega\Lambda$ is given by $\sum_j \Omega_{ij}\Lambda_{jk}$.

j) Show that if $\Omega|\phi\rangle = |\phi'\rangle$ implies $\langle\phi|\Omega = \langle\phi'|$ for all kets, then $\Omega_{ij} = (\Omega_{ji})^*$. What name is given to operators satisfying this condition?

k) By using the resolution of the identity, $I = \sum_i |i\rangle\langle i|$, show that $|\phi\rangle = \sum_i |i\rangle\langle i|\phi\rangle$.

5) Orthogonalisation of bases – optional: The three vectors, $|V_1\rangle$, $|V_2\rangle$, and $|V_3\rangle$ are elements of the three-dimensional linear vector space $\mathcal{V}^{(3)}(\mathcal{C})$. Their inner products are given by the matrix elements

$$\begin{pmatrix} \langle V_1|V_1\rangle & \langle V_1|V_2\rangle & \langle V_1|V_3\rangle \\ \langle V_2|V_1\rangle & \langle V_2|V_2\rangle & \langle V_2|V_3\rangle \\ \langle V_3|V_1\rangle & \langle V_3|V_2\rangle & \langle V_3|V_3\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \\ i/\sqrt{2} & 0 & \sqrt{2} \end{pmatrix}.$$

a) Is the set $\{|V_i\rangle\}$ linearly independent?

b) Use the Gram-Schmidt construction (Shankar, page 18 **) to build an orthonormal basis $\{|i\rangle\}$ in terms of $\{|V_i\rangle\}$.

c) If the new basis, $\{|i\rangle\}$, is given in terms of the old basis, $\{|V_i\rangle\}$, by the equation $|i\rangle = \sum_j M_{ji}|V_j\rangle$, then what are the elements of the matrix M_{ji}?

d) In part (c) the matrix M_{ji} connects two bases. Is M_{ji} a unitary matrix?

e) If M_{ji} is not unitary, what property of the set $\{|V_i\rangle\}$ is responsible for this?

6) Matrices and commutators: The operator M has matrix elements

$$\begin{pmatrix} \langle 1|M|1\rangle & \langle 1|M|2\rangle \\ \langle 2|M|1\rangle & \langle 2|M|2\rangle \end{pmatrix} = \begin{pmatrix} 2 & i \\ 0 & 1+i \end{pmatrix}.$$

The operator N has matrix elements

$$\begin{pmatrix} \langle 1|N|1\rangle & \langle 1|N|2\rangle \\ \langle 2|N|1\rangle & \langle 2|N|2\rangle \end{pmatrix} = \begin{pmatrix} i & i \\ 0 & 1 \end{pmatrix}.$$

a) Compute the matrix elements of the operator MN?
b) Compute the matrix elements of the operator NM?

The commutator of two operators, R and S, is denoted $[R, S]$, and is defined to be the operator $RS - SR$.

c) Compute the matrix elements of the commutator $[M, N]$?
d) Compute the matrix elements of $M^\dagger$, of $N^\dagger$, and of $M^\dagger N^\dagger$.
e) Do the matrix elements of $M^\dagger N^\dagger$ equal the matrix elements of $(MN)^\dagger$?
f) Write down the rule for the adjoint of a product of two operators?

The matrix $L_{ij} \equiv \langle i|L|j\rangle$ represents the operator L. The components $\phi_i \equiv \langle i|\phi\rangle$ represent the ket vector $|\phi\rangle$.

g) Write down the components of the ket vector $L|\phi\rangle$, in terms of L_{ij} and ϕ_i?
h) Write down the components of the ket vector $L^\dagger|\phi\rangle$, in terms of L_{ij} and ϕ_i?
i) Write down the components of the bra vector $\langle\phi|$ in terms of the components ϕ_i?
j) Write down the components of the bra vector $\langle\phi|L$?
k) Write down the components of the bra vector $\langle\phi|L^\dagger$?

In terms of the matrix elements of M and N given explicitly above, and the components, ϕ_i, of the ket $|\phi\rangle$, compute:

l) the components of the ket $M|\phi\rangle$?
m) the components of the ket $M^\dagger|\phi\rangle$?
n) the components of the ket $MN|\phi\rangle$?

7) Exercises from Shankar – optional:

c) Exercise 1.8.3 on page 41.
d) Exercise 1.8.6 on page 42.