Physics 580 | **Quantum Mechanics I** | Prof. P. M. Goldbart
Handout 7 | `webusers.physics.illinois.edu/~goldbart/` | 3135 (& 2115) ESB
14 September 2010 | Homework 3 | University of Illinois

1) Schwarz inequality; triangle inequality: Let $|S\rangle$ and $|T\rangle$ be two vectors.

a) By considering the positivity of the norm of the vector

$$|S\rangle - \frac{\langle T|S\rangle}{\langle T|T\rangle}|T\rangle$$

establish the validity of the Schwarz inequality, i.e., $\langle S|S\rangle\langle T|T\rangle \geq |\langle S|T\rangle|^2$.

b) Give, and describe the origin of, the condition under which the inequality becomes an equality.

c) By considering the norm of the vector $|S\rangle + |T\rangle$, and then using the Schwarz inequality, prove the triangle inequality

$$\sqrt{\{\langle S| + \langle T|\}\{|S\rangle + |T\rangle\}} \equiv |S+T| \leq |S| + |T| \equiv \sqrt{\langle S|S\rangle} + \sqrt{\langle T|T\rangle}\,.$$

d) Give, and describe the origin of, the condition under which the inequality becomes an equality.

2) Bras as linear functionals: The two-dimensional linear vector space over the real numbers $\mathcal{V}^{(2)}(\mathcal{R})$ is spanned by the two orthonormal basis vectors $|1\rangle$ and $|2\rangle$.

a) What are the values of the elements of the matrix of inner products

$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|2\rangle \\ \langle 2|1\rangle & \langle 2|2\rangle \end{pmatrix}\ ?$$

An arbitrary ket $|\psi\rangle$ is a linear combination of these basis vectors: $|\psi\rangle = |1\rangle a_1 + |2\rangle a_2$.

b) Show that the appropriate expansion coefficients, a_1 and a_2, are given by $a_1 = \langle 1|\psi\rangle$ and $a_2 = \langle 2|\psi\rangle$.

Recall that a linear functional is an object $\langle\phi|$, called a bra, that acts on a ket $|\theta\rangle$ and returns the scalar $\langle\phi|\theta\rangle$. It is natural to choose as a basis for the set of linear functionals on $\mathcal{V}^{(2)}(\mathcal{R})$ the bras $\langle 1|$ and $\langle 2|$. For example, when we act with $\langle 1|$ on $|\theta\rangle$ the result is $\langle 1|\theta\rangle$. A general bra may be expanded in terms of the basis as $\langle\phi| = b_1\langle 1| + b_2\langle 2|$.

c) Show that the correct expansion coefficients, b_1 and b_2, are given by $b_1 = \langle\phi|1\rangle$ and $b_2 = \langle\phi|2\rangle$.

d) Hence show that, together, the set of expansion coefficients $\{\langle i|\psi\rangle\}_{i=1}^{2}$ (for the ket $|\psi\rangle$) and the set of expansion coefficients $\{\langle\phi|i\rangle\}_{i=1}^{2}$ (for the bra $\langle\phi|$) are sufficient to calculate the scalar $\langle\phi|\psi\rangle$ that results from the action of the bra $\langle\phi|$ on the ket $|\psi\rangle$.

3) A linear vector space: Consider the linear vector space (LVS) $\mathcal{V}^{(2)}(\mathcal{C})$.

a) What is the dimension of this LVS?
b) From what field do the scalars come?
c) Suppose you are given the basis: $\{|1\rangle, |2\rangle\}$. Write down an expression for an arbitrary ket $|\phi\rangle$ in terms of this basis.

Suppose that the inner products between basis kets are given by

$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|2\rangle \\ \langle 2|1\rangle & \langle 2|2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

d) Are the basis kets normalised?
e) Is the basis orthogonal?
f) For your arbitrary ket in part (c), calculate the value of $\langle 1|\phi\rangle$.
g) For your arbitrary ket in part (c), calculate the value of $\langle 2|\phi\rangle$.
h) If your arbitrary ket $|\phi\rangle$ is $\alpha|1\rangle + \beta|2\rangle$, why are your answers to parts (f) and (g) *not* α and β?
i) Calculate the value of $\langle\phi|\phi\rangle$ in terms of α and β.
j) Write down a normalised version $|\tilde{\phi}\rangle$ of the ket $|\phi\rangle$
k) Consider a second arbitrary ket $|\psi\rangle = \gamma|1\rangle + \delta|2\rangle$. Compute $\langle\psi|\phi\rangle$.
l) Why is $\langle\psi|\phi\rangle$ *not* equal to $\gamma^*\alpha + \delta^*\beta$?
m) Calculate $|\psi\rangle + |\phi\rangle$ in terms of basis kets.
n) Calculate $|\psi\rangle + i|\phi\rangle$ in terms of basis kets.
o) Use the Gram-Schmidt orthogonalisation procedure, and the process of normalisation, to construct an orthonormal basis, $\{|a_1\rangle, |a_2\rangle\}$ from the basis $\{|1\rangle, |2\rangle\}$. Start with $|a_1\rangle \propto |1\rangle$.
p) Is your resulting orthonormal basis unique?
q) Express $|\phi\rangle$ in the orthonormal basis, in terms of α and β.
r) Express $|\psi\rangle$ in the orthonormal basis, in terms of γ and δ.
s) Calculate $\langle\psi|\phi\rangle$ using the orthonormal basis.
t) Does this agree with your calculation in part (k)?

4) Dirac delta function – optional: In this question we shall explore some of the properties of the Dirac delta function (the generalisation to the continuum of the Kronecker delta symbol). The Dirac delta function $\delta(x)$ has as its argument the real variable x. It has the following rather striking properties:

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty} dx\, \delta(x) = 1.$$

From these properties you can see that one way to think of $\delta(x)$ is that it has an infinitely high spike at the origin, is zero elsewhere, and has unit area.

a) Give a heuristic (*i.e.*, sloppy) proof that, for sufficiently smooth $f(x)$,

$$\int_{-\infty}^{\infty} dx\, \delta(x)\, f(x) = f(0).$$

b) Show that

$$\int_{-\infty}^{\infty} dx\, \delta(x-a)\, f(x) = f(a).$$

Often we shall neglect to write the limits on integrals.

c) By using integration by parts, show that

$$\int_{-\infty}^{\infty} dx\, \delta'(x)\, f(x) = -f'(0),$$

where $f'(x) \equiv df(x)/dx$ and $\delta'(x) \equiv d\delta(x)/dx$.

d) Consider the family of gaussian functions,

$$\Delta_\xi(x) \equiv \frac{1}{\sqrt{2\pi\xi^2}} \exp\left(-\frac{x^2}{2\xi^2}\right),$$

parametrised by the width ξ. Show that the defining properties of $\delta(x)$ are satisfied by $\lim_{\xi\to 0} \Delta_\xi(x)$ and, hence, that this limit represents $\delta(x)$.
Hint: $\int_{-\infty}^{\infty} dy\, \exp(-y^2/2) = \sqrt{2\pi}$.

e) By considering the Fourier transform,

$$\tilde{f}(q) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-iqx},$$

and its inverse,

$$f(x) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, \tilde{f}(q)\, e^{iqx},$$

establish that

$$f(y) = \int_{-\infty}^{\infty} dx\, f(x) \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iq(y-x)}.$$

Thus, demonstrate that

$$\delta(y) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqy}.$$

This integral representation of the Dirac delta function is known as the Fourier representation of the Dirac delta function.

f) By considering the equation

$$f(y) = \int_{-\infty}^{\infty} dx\, \delta(y-x)\, f(x),$$

justify the statement that the delta function is a representation of the identity operator.

g) By considering the Fourier representation, show that $\delta(ax) = |a|^{-1}\, \delta(x)$.

h) Demonstrate this result by starting with the gaussian representation of part (d).

We can also have higher dimensional delta functions, by which we mean delta functions with vector arguments. For example, if $\mathbf{r}$ is a three-dimensional cartesian vector with cartesian components x, y and z, then the delta function $\delta^{(3)}(\mathbf{r})$ is defined to be $\delta^{(3)}(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)$. Often the superscript (3) is omitted.

i) Write down the Fourier representation of $\delta^{(3)}(\mathbf{r})$ in terms of the vector $\mathbf{r}$ and an integral over the vector $\mathbf{q}$.

j) Write down a representation for $\delta^{(3)}(\mathbf{r})$ in terms of the gaussian function.

k) A real function $f(x)$ of a single variable x has zeros at the set of points $\{x_i\}$. At these zeros, the gradient of f is non-vanishing, and takes the values $\{f_i^{(1)}\}$. Show that

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f_i^{(1)}|}.$$

l) The real symmetric 3×3 matrix $\mathcal{A}$ is positive-definite. (This means that all its eigenvalues are positive real numbers.) Demonstrate, by using either the gaussian or the Fourier representation, that

$$\delta^{(3)}(\mathcal{A}\mathbf{x}) = \frac{\delta^{(3)}(\mathbf{x})}{|\mathrm{det}\mathcal{A}|}.$$

A third representation of the delta function is very convenient in the context of complex variables. Consider the real variables ω and ϵ. Follow the strategy given below to establish that

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \mathrm{Im} \frac{1}{\omega - i\epsilon} = \delta(\omega),$$

where Im denotes the imaginary part.

m.i) Show that

$$\mathrm{Im}\,\frac{1}{\omega - i\epsilon} = \frac{\epsilon}{\omega^2 + \epsilon^2}.$$

m.ii) Show that

$$\lim_{\omega \to 0} \lim_{\epsilon \to 0} \frac{\epsilon}{\omega^2 + \epsilon^2} = 0,$$

$$\lim_{\epsilon \to 0} \lim_{\omega \to 0} \frac{\epsilon}{\omega^2 + \epsilon^2} = \infty.$$

m.iii) By using the substitution $\omega = \epsilon \tan \theta$, or otherwise, show that

$$\int_{-\infty}^{\infty} d\omega \, \frac{\epsilon}{\omega^2 + \epsilon^2} = \pi.$$

m.iv) Put together these pieces to argue that, indeed,

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \mathrm{Im} \frac{1}{\omega - i\epsilon} = \delta(\omega)$$

is a representation of the delta function.