This homework is due to be handed in to the PHYS 580 Homework Box by 2:30 p.m. on Thursday September 9.

1) Lagrangian for a compound plane pendulum: Consider a pendulum comprising an upper segment, of length L_1 and bob mass M_1, and a lower segment that is freely hinged to the bob of the upper segment, of length L_2 and bob mass M_2. The segments move in a common vertical plane in the presence of gravity (acceleration g). Let the angular displacements of the upper and lower segments from the vertical direction be, respectively, θ_1 and θ_2.

a) Construct a Lagrangian for the compound plane pendulum in terms of θ_1 and θ_2.
b) Hence, obtain the equations of motion for the compound plane pendulum.
c) By linearizing about the position of mechanical equilibrium, determine the frequencies of the normal modes of oscillation of the compound pendulum in terms of its parameters,

2) Stephenson-Kapitza pendulum: This type of pendulum is a simple pendulum of length L and natural frequency ω, except that the point of support is forced to vibrate. In the simplest case, which we consider here, the point of support undergoes vertical harmonic oscillations of frequency $\Omega\omega$ and amplitude qL.

a) Construct the Lagrangian for the Stephenson-Kapitza pendulum, choosing as the coordinate the angular displacement θ away from the downward vertical.
b) Hence, obtain the equation of motion for the Stephenson-Kapitza pendulum.

What is striking about the Stephenson-Kapitza pendulum is that if the support is vibrated sufficiently rapidly (i.e., for $q\Omega > \sqrt{2}$), the pendulum has a stable configuration when it is inverted (i.e., when $\theta = \pi$), about which it can undergo stable, small oscillations at a frequency given by $\omega\sqrt{-1 + \frac{1}{2}q^2\Omega^2}$. For further details, see E.I. Butikov, *On the dynamic stabilization of an inverted pendulum*, Am. J. Phys. **69** (2001), 755-768; and/or `http://www.fam.web.mek.dtu.dk/FVP/Jon/Theory1.html`. The method of multiple-scale analysis provides a mathematically precise approach to this problem; for this and many other powerful techniques, see, e.g., C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, 1978).

3) Lagrangian for a charged particle: In this question we will obtain the equation of motion for a particle of mass m and charge e moving in an electromagnetic field prescribed by the scalar potential $\phi(\mathbf{r},t)$ and the vector potential $\mathbf{A}(\mathbf{r},t)$. Recall that the electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ can be obtained from these potentials through

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B}(\mathbf{r},t) &= \nabla\times\mathbf{A}, \end{aligned}$$

where c is the speed of light *in vacuo*. The lagrangian for this system is given by

$$\mathcal{L}(\mathbf{q},\dot{\mathbf{q}},t) = \frac{m}{2}|\dot{\mathbf{q}}|^2 - e\phi(\mathbf{q},t) + \frac{e}{c}\,\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q},t),$$

where $\mathbf{q}$ denotes the position vector and $\dot{\mathbf{q}}$ denotes the velocity vector, $d\mathbf{q}/dt$.

a) Construct the Euler-Lagrange equation for $\mathbf{q}(t)$.

b) Show that the Euler-Lagrange equation is equivalent to the Lorentz equation

$$m\ddot{\mathbf{q}}(t) = e\mathbf{E}(\mathbf{q},t) + \frac{e}{c}\dot{\mathbf{q}}\times\mathbf{B}(\mathbf{q},t).$$

c) Derive the canonical momentum conjugate to $\mathbf{q}$.

d) Build the hamiltonian, $\mathcal{H}$, in terms of ϕ and $\mathbf{A}$.

Now suppose that $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ are replaced by $\phi'(\mathbf{r},t)$ and $\mathbf{A}'(\mathbf{r},t)$, where

$$\begin{aligned} \phi'(\mathbf{r},t) &= \phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\chi(\mathbf{r},t) \\ \mathbf{A}'(\mathbf{r},t), &= \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t), \end{aligned}$$

and $\chi(\mathbf{r},t)$ is an arbitrary function of $\mathbf{r}$ and t. This is known as a gauge transformation.

e) What effect does this transformation have on $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$?

f) Show that the change in $\mathcal{L}$ resulting from this change in $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ can be written as a total time-derivative. What effect does this change have on the action? What effect does it have on the equation of motion?

4) Conservation of linear momentum: A small shift in trajectory, $q(t) \to q(t) + \epsilon f(q(t),\dot{q}(t),t)$, generates a shift in the lagrangian, $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$.

a) Show that if the shift, $\delta\mathcal{L}$, is a total time-derivative, *i.e.*,

$$\delta\mathcal{L} = \epsilon\frac{d}{dt}B(q(t),\dot{q}(t),t),$$

then the quantity $B - f(\partial\mathcal{L}/\partial\dot{q})$ is conserved.

b) Use this result to demonstrate that if the lagrangian is translationally invariant, *i.e.*, $\partial\mathcal{L}/\partial q = 0$, then linear momentum is conserved.

5) Legendre transformations – optional: In this question we will examine Legendre transformations and work out an example from thermodynamics. Consider a function of a single variable, $f(x)$. Construct from it a new function $y(x) \equiv df/dx$. Now introduce a new function, the Legendre transformation of f, called g. The function g is defined by

$$g(y) = -f(x(y)) + y\, x(y).$$

a) Show that $dg/dy = x(y)$.

Consider the Helmholtz free energy, $F(T, H)$, for a magnetic system at temperature T in an external field H. The magnetisation, $M(T, H)$ is given by

$$M(T, H) = -\frac{\partial F}{\partial H}.$$

The Gibbs free energy, $G(T, M)$ is defined to be the Legendre transformation of F with respect to the variable H:

$$G(T, M) = F(T, H(T, M)) + M\, H(T, M).$$

b) Show that the equation of state is given by

$$H(T, M) = \frac{\partial}{\partial M}\, G(T, M).$$

6) More on cartesian vectors – optional: The purpose of this question is two-fold. Firstly, we will investigate some of the properties of the vector product, denoted $\times$, and the related differential operator, curl, denoted $\nabla\times$. Secondly, we will solve the problems using summation convention so that we get some more practice with it. As with *Homework 0*, we consider an orthonormal basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, for 3-dimensional cartesian vectors, $\mathbf{x}$. The basis is said to be right-handed because

$$\begin{aligned} \mathbf{e}_1\times\mathbf{e}_2 &= \mathbf{e}_3 \\ \mathbf{e}_2\times\mathbf{e}_3 &= \mathbf{e}_1, \\ \mathbf{e}_3\times\mathbf{e}_1 &= \mathbf{e}_2. \end{aligned}$$

We can express these relationships much more compactly using the symbol $\epsilon_{\mu\nu\rho}$, known as the Levi-Civita symbol, or the antisymmetric third-rank tensor. This tensor takes on the following values in all cartesian coordinate systems:

$$\epsilon_{\mu\nu\rho} = \begin{cases} +1, & \text{if } \mu\nu\rho = 123, 231, \text{ or } 312; \\ -1, & \text{if } \mu\nu\rho = 132, 213, \text{ or } 321; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\epsilon_{\mu\nu\rho}$ is totally antisymmetric, *i.e.*, its value changes sign whenever any pair of indices are exchanged, *e.g.*, $\epsilon_{123} = -\epsilon_{213} = 1$. This requirement forces $\epsilon_{\mu\nu\rho}$ to vanish whenever two or more of its indices are the same, *e.g.*, $\epsilon_{113} = 0$. This property is extremely useful, as we shall see, when it comes to proving certain results involving vector products and the curl operator.

In terms of $\epsilon_{\mu\nu\rho}$, the vector products between basis vectors become

$$\mathbf{e}_\mu \times \mathbf{e}_\nu = \epsilon_{\mu\nu\rho}\, \mathbf{e}_\rho,$$

where the implied summation on ρ recovers the previously-given results for the cases $\mu \neq \nu$, and also includes results when $\mu = \nu$. Starting with these definitions, and the results from last week's homework if necessary, verify the following statements using summation convention:

a.1) $\epsilon_{\mu\nu\rho}\, \epsilon_{\mu\sigma\tau} = \delta_{\nu\sigma}\, \delta_{\rho\tau} - \delta_{\nu\tau}\, \delta_{\rho\sigma}$
a.2) $\epsilon_{\mu\nu\rho}\, \epsilon_{\mu\nu\tau} = 2\, \delta_{\rho\tau}$
a.3) $\epsilon_{\mu\nu\rho}\, \epsilon_{\mu\nu\rho} = 6$
a.4) $\mathbf{A}\times\mathbf{B} = A_\mu\, B_\nu\, \epsilon_{\mu\nu\rho}\, \mathbf{e}_\rho$
a.5) $(\mathbf{A}\times\mathbf{B})_\rho = A_\mu\, B_\nu\, \epsilon_{\mu\nu\rho}$
a.6) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \epsilon_{\mu\nu\rho}\, A_\mu\, B_\nu\, C_\rho$
a.7) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$
a.8) $\mathbf{A}\times\mathbf{A} = \mathbf{0}$
a.9) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

Now consider scalar and vector fields, *i.e.*, scalar-valued functions, $f(\mathbf{x})$, and vector-valued functions, $\mathbf{g}(\mathbf{x}) = \mathbf{e}_\mu\, g_\mu(\mathbf{x})$, of a position vector, $\mathbf{x}$. The curl operator, $\nabla\times$, operates on a vector field, $\mathbf{g}(\mathbf{x})$ to produce new vector field, denoted $\nabla\times\mathbf{g}(\mathbf{x})$. It is defined in the following way:

$$\nabla\times\mathbf{g}(\mathbf{x}) \equiv \sum_{\mu,\nu,\rho=1}^{3} \mathbf{e}_\mu\, \epsilon_{\mu\nu\rho}\, \frac{\partial}{\partial x_\nu} g_\rho(\mathbf{x}) = \mathbf{e}_\mu\, \epsilon_{\mu\nu\rho}\, \frac{\partial}{\partial x_\nu}\, g_\rho(\mathbf{x}) = \mathbf{e}_\mu\, \epsilon_{\mu\nu\rho}\, \partial_\nu\, g_\rho(\mathbf{x}).$$

Using these definitions, verify the following statements:

b.1) $\nabla\times\mathbf{x} = \mathbf{0}$
b.2) $\nabla\times(\mathbf{H}\times\mathbf{x}) = 2\,\mathbf{H}$, for constant $\mathbf{H}$
b.3) $\nabla\cdot(\nabla\times\mathbf{g}(\mathbf{x})) = 0$
b.4) $\nabla\times(\nabla f(\mathbf{x})) = \mathbf{0}$
b.5) $\nabla\times(f(\mathbf{x})\,\mathbf{g}(\mathbf{x})) = f\,\nabla\times\mathbf{g} + (\nabla f)\times\mathbf{g}$
b.6) $\nabla\times(\mathbf{g}(\mathbf{x})\times\mathbf{h}(\mathbf{x})) = \mathbf{g}\,\nabla\cdot\mathbf{h} - \mathbf{h}\,\nabla\cdot\mathbf{g} + (\mathbf{h}\cdot\nabla)\,\mathbf{g} - (\mathbf{g}\cdot\nabla)\,\mathbf{h}$

7) Wave mechanics – optional: Note that this question is not intended for handing in and grading. However, please come and talk to me if it contains unfamiliar ideas and you feel uncertain about how to answer any parts.

The ground-state wave function of a one-dimensional quantum mechanical system of mass m is given by

$$\psi(x) = N \exp\Big\{-\frac{m\omega x^2}{2\hbar} + \frac{m\omega x a}{\hbar}\Big\}$$

where N, ω and a are constants.

a) If $\psi(x)$ represents a normalised wave function, calculate N.
b) Is your choice of N unique?
c) To within a constant, what is the external potential which the particle experiences?
d) To within a constant, what is the ground state energy?
e) Write down the probability distribution for the position of the particle?
f) Calculate its mean position.
g) Calculate its root-mean-square position, δx.
h) The momentum eigenfunctions are given by

$$\phi_p = \frac{1}{\sqrt{2\pi\hbar}} \exp\Big\{\frac{ipx}{\hbar}\Big\}.$$

Compute the wave function in momentum space.
i) Write down the probability distribution in momentum space.
j) Calculate the mean momentum of the particle.
k) Calculate its root-mean-square momentum, δp.
l) What is the value of the product $\delta p\, \delta x$?
m) Is there a wave function which gives rise to a smaller value of the product $\delta p\, \delta x$?
n) Are there wave functions for which $\delta p\, \delta x = 0$?

Note: $\int_{\infty}^{\infty} dy \exp\{-y^2/2 - iqy\} = \sqrt{2\pi} \exp\{-q^2/2\}$.