TWO-QUBIT MIXED STATES AND THE ENTANGLEMENT-ENTROPY FRONTIER

TZU-CHIEH WEI[1], KAE NEMOTO[2], PAUL M. GOLDBART[1],
PAUL G. KWIAT[1], WILLIAM J. MUNRO[3], AND FRANK VERSTRAETE[4]

[1]*Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080, USA*

[2]*Informatics, Bangor University, Bangor LL57 1UT, UK*

[3]*Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol, BS34 SQ2, UK*

[4]*Department of Mathematical Physics and Astronomy, Ghent University, Belgium*

Maximally entangled mixed states are states that, for a given mixedness (entropy), achieve the greatest possible entanglement. For two-qubit systems and for various combinations of entanglement and mixedness measures, we determine the form of the corresponding maximally entangled mixed states analytically. We show that their forms can vary with the combination of entanglement and mixedness measures chosen. Moreover, for certain combinations, the forms of the maximally entangled mixed states can change discontinuously at a specific value of the entropy.

1 Introduction

Over the past decade, the physical characteristic of the entanglement of quantum states has been recognized as a central resource in various aspects of quantum information processing.[1] Given the central status of entanglement, the task of quantifying the degree to which a state is entangled is important, and several measures have been proposed to quantify it.[2] It is worth remarking that even for the smallest Hilbert space capable of exhibiting entanglement, i.e., the two-qubit system, there are aspects of entanglement that remain to be explored.

Among the family of mixed quantum mechanical states, special status should be accorded to those that, for a given value of the entropy (or mixedness), have the largest possible degree of entanglement.[3] The reason for this is that such states can be regarded as mixed-state generalizations of Bell states, the latter being known to be the maximally entangled two-qubit *pure* states. Hence, this kind of *mixed* states could be expected to provide useful resources for quantum information processing.

In order to construct maximally entangled mixed states (MEMS), one needs to define and quantify the notions of both entanglement and mixedness. Yet, different entanglement measures can lead to different orderings of entangled states.[4] There can also be ordering problems for mixedness. These two ordering problems imply that what states constitute MEMS may depend on the measures of both entanglement and mixedness.

The primary objective of this Paper is to determine the *frontiers*, i.e., the boundaries of the regions occupied by physically allowed states in these planes, and to identify the structure of these MEMS under various measures.

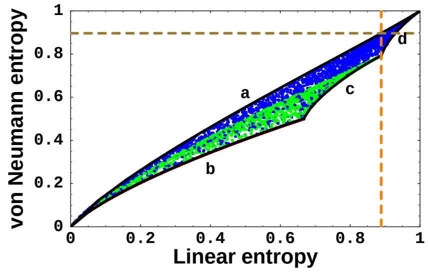

Figure 1. Comparison of two entropies. Dots represent randomly generated states; the segments a, b, c, and d constitute the boundary of physically-allowed states. The figure shows that, e.g., there exist mixed states ρ_1 and ρ_2 for which $S_{\rm V}(\rho_1) > S_{\rm V}(\rho_2)$ whereas $S_{\rm L}(\rho_1) < S_{\rm L}(\rho_2)$. Dashed lines indicate thresholds of entropies beyond which no states can be entangled.

2 Entanglement and mixedness

In this section we briefly review measures of entanglement (entanglement of formation and negativity) and mixedness (linear and von Neumann entropies) that we consider.

The entanglement of formation for a mixed state ρ is defined as the minimal average number of maximally entangled pure states consumed in order to realize the ensemble described by ρ, i.e.,

$$E_{\rm F}(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i\, E(|\psi_i\rangle), \quad \text{with} \quad E(|\psi_i\rangle) \equiv -{\rm Tr}\,\rho_{\rm B} \log(\rho_{\rm B}), \tag{1}$$

where $\rho_{\rm B} \equiv {\rm Tr}_{\rm A}|\psi_i\rangle\langle\psi_i|$ and the minimization is taken over $\{p_i\}$ and $\{|\psi_i\rangle\}$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. We remark that $E_{\rm F}$ is, in general, difficult to calculate for mixed states. Fortunately, for the two-qubit case Wootters[5] has derived a closed form for $E_{\rm F}$.

Negativity is an indication of the extent to which a state violates the positive partial transpose (PPT) criterion[6] of separability. It is defined as twice the absolute value of the sum of the negative eigenvalues of the partially transposed matrix $\rho_{\rm B}^{\rm T}$. In two-qubit systems it can be shown that the partial transpose of a density matrix can have at most one negative eigenvalue[8] and, hence, $N(\rho) = 2\max(0, -\lambda_4)$, where λ_4 is the lowest eigenvalue of $\rho^{{\rm T}_B}$. Even though $N = 0$ is not, in general, a sufficient condition for separability (except for two-qubit and qubit-qutrit systems), it is readily computable. We remark that these two measures ($E_{\rm F}$ and N) can give different orderings for different pairs of states.[4]

The two mixedness measures we consider are linear entropy $S_{\rm L}$ and von Neumann entropy $S_{\rm V}$, defined in two-qubit systems via

$$S_{\rm L}(\rho) \equiv \frac{4}{3}(1 - {\rm Tr}\rho^2), \quad S_{\rm V}(\rho) \equiv -{\rm Tr}(\rho \log_4 \rho); \tag{2}$$

they are properly normalized in the range $[0, 1]$. We illustrate the ordering problem of these two mixedness measures in Fig. 1.

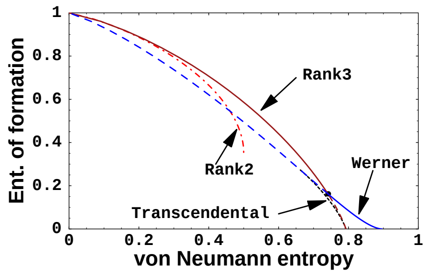

Figure 2. Entanglement frontier: $E_{\rm F}$ vs. $S_{\rm V}$. The physically-allowed region is below the solid curve, i.e., the frontier. The frontier consists of two branches: that given by $\rho_{\rm i}$ (indicated by "Rank 3") and that given by $\rho_{\rm ii}$ (Werner states). Two additional curves indicated by "Rank 2" and "Transcendental" (not shown) are from those states that also satisfy the stationarity conditions but are not globally maximal.

3 Entanglement-versus-mixedness frontiers

In this section, we describe how to derive and present the forms of MEMS under various measures. Employing the results of Verstraete et al.,[7] it can be shown that the states with maximal $E_{\rm F}$ or N can be parametrized in the following way[8] (up to local unitary transformations):

$$\rho = \begin{bmatrix} x+\frac{r}{2} & 0 & 0 & \frac{r}{2} \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ \frac{r}{2} & 0 & 0 & x+\frac{r}{2} \end{bmatrix}, \quad \text{with } x,a,b,r\geq 0 \text{ and } {\rm Tr}(\rho)=1. \tag{3}$$

Straightforward calculations show that

$$E_{\rm F} = h\big([1+\sqrt{1-C^2}]/2\big), \quad \text{with } C\equiv \max\{r-2\sqrt{ab},0\}, \tag{4a}$$

$$N = \max\{\sqrt{(a-b)^2+r^2}-(a+b),0\}, \tag{4b}$$

where $h(x)\equiv -x\log_2(x)-(1-x)\log_2(1-x)$.

By maximizing entanglement[a] at fixed entropy, we can then derive the forms of MEMS (frontier states) under the four combinations of entanglement and mixedness. We list the results as follows.

I. $E_{\rm F}$ vs. $S_{\rm L}$ frontier states. The form of MEMS in this case was derived by Munro et al.[3] and shown to consist of two branches. At the point where the two branches meet, the forms of states on the two sides coincide.

II. N vs. $S_{\rm L}$ frontier states. There are two inequivalent families of MEMS which give the same frontier:

$$\rho^{(1)}_{{\rm MEMS}:N,S_{\rm L}}(r)=\rho_{\rm Werner}(r), \quad \rho^{(2)}_{{\rm MEMS}:N,S_{\rm L}}(r)=\begin{bmatrix} \frac{1+\sqrt{3r^2+1}}{6} & 0 & 0 & \frac{r}{2} \\ 0 & \frac{4-2\sqrt{3r^2+1}}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r}{2} & 0 & 0 & \frac{1+\sqrt{3r^2+1}}{6} \end{bmatrix}, \tag{5}$$

[a] For the case of $E_{\rm F}$, it is equivalent and convenient to maximize C instead.

where the Werner state $\rho_{\rm Werner}(r)$ is defined to be $r|\phi^+\rangle\langle\phi^+| + \frac{1-r}{4}I$.

III. $E_{\rm F}$ *vs.* $S_{\rm V}$ *frontier states.* The frontier states also consist of two branches:

$$\rho_{{\rm MEMS};E_{\rm F},S_{\rm V}} = \begin{cases} \rho_{\rm ii}(C), & \text{for } 0 \le C \le C^*; \\ \rho_{\rm i}(C), & \text{for } C^* \le C \le 1; \end{cases} \quad \text{with } C^* \approx 0.741, \tag{6a}$$

$$\rho_{\rm i}(C) = \begin{bmatrix} \frac{4-\sqrt{4-3C^2}}{6} & 0 & 0 & \frac{C}{2} \\ 0 & \frac{\sqrt{4-3C^2}-1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{C}{2} & 0 & 0 & \frac{4-\sqrt{4-3C^2}}{6} \end{bmatrix}, \quad \rho_{\rm ii}(C) = \begin{bmatrix} \frac{2+C}{6} & 0 & 0 & \frac{1+2C}{6} \\ 0 & \frac{1-C}{6} & 0 & 0 \\ 0 & 0 & \frac{1-C}{6} & 0 \\ \frac{1+2C}{6} & 0 & 0 & \frac{2+C}{6} \end{bmatrix}. \tag{6b}$$

Note that at $C = C^*$ the density matrix on each branch can never be the same, and, hence, as one moves across $C = C^*$ along the frontier, the state changes discontinously. Moreover, $\rho_{\rm ii}$ is in fact a Werner state: $\rho_{\rm ii}(C) = \rho_{\rm Werner}(r)\big|_{r=(2C+1)/3}$.

IV. N vs. $S_{\rm V}$ *frontier states.* In this case the frontier states are Werner states.

We illustrate the entanglement vs. mixedness plane for the case of $E_{\rm F}$ vs. $S_{\rm V}$ in Fig. 2.

4 Concluding remarks

We have determined families of maximally entangled mixed states (MEMS) and boundary of the physically-allowed region in the entanglement-mixedness plane (frontier) under various entanglement and mixedness measures. The frontier states under $E_{\rm F}$ vs. $S_{\rm V}$ behave *discontinously* at a specific point on the frontier. What precise resources the various MEMS furnish remains an open question.

This work was supported in part by NSF, DOE, QUIPROCONE, and EQUIP.

References

1. See, e.g., M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
2. For a review, see M. Horodecki, Quan. Info. and Comp. **1**, 3 (2001).
3. W. J. Munro et al., Phys. Rev. A **64**, 030302 (2001).
4. J. Eisert and M. B. Plenio, J. Mod. Opt. **46**, 145 (1999); F. Verstraete et al., J. Phys. A **34**, 10327 (2001); S. Virmani and M. B. Plenio, Phys. Lett. A **268**, 31 (2000).
5. W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
6. A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki et al., Phys. Lett. A **223**, 1 (1996).
7. F. Verstraete et al., Phys. Rev. A **64**, 012316 (2001).
8. T. C. Wei et al., quant-ph/0208138.