Quantifying multipartite entanglement

Tzu-Chieh Wei*, Joseph B. Altepeter*, Dyutiman Das*, Marie Ericsson†*, Paul M. Goldbart*, Swagatam Mukhopadyay*, William J. Munro** and Smitha Vishveshwara*

*Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080, USA
†Centre for Quantum Computation, DAMTP, University of Cambridge, Cambridge CB3 0WA, UK
**Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol, BS34 SQ2, UK

Abstract. A natural way of quantifying the degree of entanglement for a pure quantum state is to compare how far this state is from the set of all unentangled pure states. This geometric measure of entanglement is explored for bipartite and multipartite pure and mixed states. It is determined analytically for arbitrary two-qubit mixed states and for generalized Werner and isotropic states. It is also applied to certain multipartite mixed states, including two multipartite bound entangled states discovered by Smolin and Dür. Moreover, the geometric measure of entanglement is applied to the ground state of the Ising model in a transverse magnetic field. From this model the entanglement is shown to exhibit singular behavior at the quantum critical point.

INTRODUCTION

The notion of entanglement has been studied extensively and the entanglement itself has been recognized as a central resource for quantum information processing [1]. Much progress has been made in the bipartite setting; see, e.g., [2]. Entanglement in the multipartite setting turns out much richer and more delicate to characterize. In this paper we describe a simple measure to quantify entanglement, applicable to both bi- and multipartite settings, and examine the measure in several examples.

GEOMETRIC MEASURE OF ENTANGLEMENT

The idea of quantifying the degree of entanglement for a pure quantum state via comparing how far this state is from the set of all unentangled states has appeared in several places previously [3, 4, 5, 6]. Let us start with a multipartite system comprising n parts, each of which can have a distinct Hilbert space. Consider a general n-partite pure state (expanded in the local bases $\{|e^{(i)}_{p_i}\}$):

$$|\psi\rangle = \sum_{p_1\cdots p_n} \chi_{p_1 p_2\cdots p_n} |e^{(1)}_{p_1} e^{(2)}_{p_2} \cdots e^{(n)}_{p_n}\rangle. \tag{1}$$

We are interested in quantifying the entanglement of the system in a *global* way, and thus it is natural to compare this state to the set of pure states that are completely unentangled among any two or more of the parties:

$$|\phi\rangle \equiv \bigotimes_{i=1}^{n} |\phi^{(i)}\rangle = |\phi^{(1)}\rangle \otimes |\phi^{(2)}\rangle \otimes \ldots \otimes |\phi^{(n)}\rangle. \tag{2}$$

The maximal overlap

$$\Lambda_{\max}(\psi) \equiv \max_{\phi} |\langle\psi|\phi\rangle| \tag{3}$$

signifies the entanglement of $|\psi\rangle$; the larger $\Lambda_{\max}$ is, the less entangled $|\psi\rangle$ is.

If one is interested in entanglement between two groups of the parties, e.g., $\{1,\ldots,k\}$ and $\{k+1,\ldots,n\}$, one can replace the unentangled state $|\phi\rangle$ in Eq. (3) by

$$|\phi\rangle = |\phi^{(1\ldots k)}\rangle \otimes |\phi^{(k+1\ldots n)}\rangle, \tag{4}$$

where $|\phi^{(1\ldots k)}\rangle$ ($|\phi^{(k+1\ldots n)}\rangle$) is any state comprising parties $\{1,\ldots,k\}$ ($\{k+1,\ldots,n\}$). $\Lambda_{\max}$ then reflects the entanglement between the two groups. Other partitions can be considered. But throughout the present paper we shall be concerned with the entanglement defined globally. The two particular forms of entanglement that we shall use are

$$E_{\sin^2}(\psi) \equiv 1 - \Lambda_{\max}(\psi)^2, \text{ and } E_{\log_2}(\psi) \equiv -2\log_2 \Lambda_{\max}(\psi). \tag{5}$$

The former, $E_{\sin^2}$, is bounded above by unity, and is suitable for finite numbers of parties, whereas the latter $E_{\log_2}$, unbounded above, is applicable to arbitrary numbers of parties, including the thermodynamic limit $n \to \infty$. Moreover, via $E_{\log_2}$ one can also address the question of entanglement per party by defining an entanglement density

$$\mathscr{E} \equiv E_{\log_2}/n, \tag{6}$$

which we shall use to calculate the entanglement density of an infinite chain of spins. $E_{\log_2}$ also has the useful feature that it gives a lower bound on the relative entropy of entanglement E_{R} [7], i.e., $E_{\log_2} \le E_{\mathrm{R}}$. Cases have been found for which the inequality becomes an equality; see Ref. [8]. We remark that the problem of determining the maximal overlap has recently been formulated in terms of *convex optimization* [9].

Given the definition of entanglement for pure states, just formulated, the extension to mixed states ρ can be built upon pure states via the *convex hull* construction, as was done for the entanglement of formation (EF) (see Ref. [10]). The essence is a minimization over all decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ into pure states:

$$E(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_{\mathrm{pure}}(\psi_i). \tag{7}$$

This convex hull construction ensures that the measure gives zero for separable states; however, in general it also complicates the task of determining mixed-state entanglement. In Ref. [6], $E_{\sin^2}$ was shown to be an entanglement monotone, i.e., it is not increasing under local operations and classical communication.

ANALYTICAL RESULTS

In this section we catalogue analytical results of the geometric measure for several illustrative examples.

Two qubits. For two qubits we can employ the results of Wootters [10] and obtain the corresponding results for the geometric measure for arbitrary two qubits [11, 6]:

$$E_{\sin^2}(\rho) = \frac{1}{2}\Big(1 - \sqrt{1 - C(\rho)^2}\Big), \tag{8}$$

where $C(\rho)$ is the Wootters concurrence of the state ρ.

Generalized Werner states. The generalized Werner states of $d \otimes d$ bipartite system are states that are invariant under

$$\mathbf{P}_1 : \rho \to \int dU\,(U\otimes U)\rho\,(U^\dagger \otimes U^\dagger), \tag{9}$$

where U is a unitary transformation. These states can be expressed as

$$\rho_{\mathrm{W}}(f) = \frac{d^2 - fd}{d^4 - d^2}\,\hat{I} + \frac{fd^2 - d}{d^4 - d^2}\,\hat{F}, \tag{10}$$

where $\hat{I}$ is the identity and $\hat{F} \equiv \sum_{ij} |ij\rangle\langle ji|$ is the swap operator. By applying to $E_{\sin^2}$ the technique of Vollbrecht and Werner for EF [12], we arrive at the geometric entanglement function for Werner states [6]:

$$E_{\sin^2}(\rho_{\mathrm{W}}(f)) = \frac{1}{2}\left(1 - \sqrt{1-f^2}\right) \quad \text{for } f \le 0, \text{ and zero otherwise.} \tag{11}$$

Isotropic states. The isotropic states are invariant under

$$\mathbf{P}_2 : \rho \rightarrow \int dU \, (U \otimes U^*)\rho \, (U^\dagger \otimes U^{*\dagger}), \tag{12}$$

and can be expressed as

$$\rho_{\rm iso}(F) \equiv \frac{1-F}{d^2-1} \left(I - |\Phi^+\rangle\langle\Phi^+| \right) + F|\Phi^+\rangle\langle\Phi^+|, \tag{13}$$

where $|\Phi^+\rangle \equiv \sum_{i=1}^{d} |ii\rangle/\sqrt{d}$ and $F \in [0,1]$. Corresponding to the results of EF for these states [13], we have [6]

$$E_{\sin^2}(\rho_{\rm iso}(F)) = 1 - \frac{1}{d}\left(\sqrt{F} + \sqrt{(1-F)(d-1)}\right)^2, \quad \text{for } 1 \ge F \ge 1/d, \text{ and zero otherwise.} \tag{14}$$

GHZ state, W state and mixtures of them. For the state $|\mathrm{GHZ}\rangle \equiv (|000\rangle + |111\rangle)/\sqrt{2}$ one can readily compute $\Lambda_{\max}$, finding that $\Lambda_{\max}(\mathrm{GHZ}) = 1/\sqrt{2}$. For the state $|\mathrm{W}\rangle \equiv (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ one finds that $\Lambda_{\max}(\mathrm{W}) = 2/3$. These results show that according to the geometric measure, the W state has higher entanglement than the GHZ state does. In fact, for these two states $E_{\rm R} = 2\log_2 \Lambda_{\max}$ [8], i.e., $E_{\rm R}(GHZ) = 1$, whereas $E_{\rm R}(W) = \log_2(9/4)$. One can actually go beyond pure states and consider, e.g., the mixture

$$x|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + y|\mathrm{W}\rangle\langle\mathrm{W}| + (1-x-y)|\widetilde{\mathrm{W}}\rangle\langle\widetilde{\mathrm{W}}|, \tag{15}$$

where $|\widetilde{\mathrm{W}}\rangle$ is obtained from $|\mathrm{W}\rangle$ by exchanging 0 and 1. This mixture, as well as other mixtures of highly symmetric, multipartite states have been studied in detail in Ref. [6].

Smolin's bound entangled state. This is a four-qubit mixed state:

$$\rho^{ABCD} \equiv \frac{1}{4}\sum_{i=0}^{3} \left(|\Psi_i\rangle\langle\Psi_i|\right)_{\rm AB} \otimes \left(|\Psi_i\rangle\langle\Psi_i|\right)_{\rm CD}, \tag{16}$$

where the $|\Psi\rangle$'s are the four Bell states $(|00\rangle \pm |11\rangle)/\sqrt{2}$ and $(|01\rangle \pm |10\rangle)/\sqrt{2}$. Although bound entangled, the amount of entanglement of Smolin's state is actually as large as that of a four-partite GHZ state. It can be shown that $E_{\sin^2}(\rho^{ABCD}) = 1/2$ and $E_{\log_2}(\rho^{ABCD}) = 1$; see Ref. [15].

Dür's bound entangled states. Dür [16] found that for $N \ge 4$ the following state is bound entangled:

$$\rho_N \equiv \frac{1}{N+1}\left(|\Psi_G\rangle\langle\Psi_G| + \frac{1}{2}\sum_{k=1}^{N}\left(P_k + \bar{P}_k\right)\right), \tag{17}$$

where $|\Psi_G\rangle \equiv \left(|0^{\otimes N}\rangle + e^{i\alpha_N}|1^{\otimes N}\rangle\right)/\sqrt{2}$ is a N-partite GHZ state; $P_k \equiv |u_k\rangle\langle u_k|$ is a projector onto the state $|u_k\rangle \equiv |0\rangle_1|0\rangle_2\ldots|1\rangle_k\ldots|0\rangle_N$; and $\bar{P}_k \equiv |v_k\rangle\langle v_k|$ projects onto $|v_k\rangle \equiv |1\rangle_1|1\rangle_2\ldots|0\rangle_k\ldots|1\rangle_N$. This state violates certain Bell inequalities for $N \ge 6$ [16, 17, 18]. In terms of the geometric measures, Dür's states can be shown to have $E_{\sin^2}(\rho_N) = 1/2(N+1)$ and $E_{\log_2}(\rho_N) = \log_2[(2N+2)/(2N+1)]$; see Ref. [15].

APPLICATION TO A QUANTUM PHASE TRANSITION

We now turn to a physical model and quantify its ground-state entanglement. The Hamiltonian $\mathscr{H}$ for the Ising model in the transverse magnetic field is given by

$$\mathscr{H} = -J\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right), \tag{18}$$

where $J > 0$ is the ferromagnetic coupling and h is the relative strength of the magnetic field to the coupling. This model exhibits a quantum phase transition at $h_c = 1$ from quantum ferromagnet ($h < 1$) to quantum paramagnet ($h > 1$). The ground state can be determined and its global entanglement (in terms of $\mathscr{E}$) can be calculated analytically [19]; the results are shown in Figs. 1. The ground-state entanglement is not maximal at the critical point, but the field-derivative of the entanglement does show divergent behavior (for an infinite chain):

$$\frac{\partial\mathscr{E}}{\partial h} \approx -\frac{1}{2\pi\ln 2}\ln|h-1|, \text{ for } |h-1| \ll 1. \tag{19}$$

This demonstrates that the singular behavior of the global entanglement can reflect the critical point of the system.

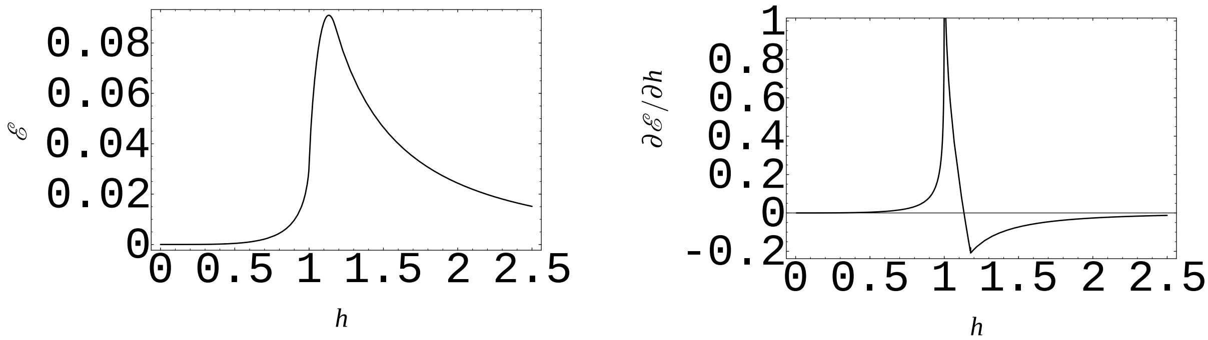

FIGURE 1. Entanglement density of the transverse Ising model for an infinite chain. Left: entanglement density vs. magnetic field. Right: h-derivative of entanglement density vs. magnetic field h.

CONCLUSIONS

We have presented a geometric measure that is suitable for quantifying entanglement among any partitions of any multiple parties of a quantum system. This measure has been determined analytically for several bi- and multipartite states. It has also been applied to the transverse Ising spin model, which exhibits a quantum phase transition. The behavior of the global entanglement density in the thermodynamic limit has been shown to display signature of the critical point.

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